Disastrous Defaults*

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Abstract

As the recent financial crisis illustrated, the default of certain entities can have disastrous effects on the economy. This paper presents a framework aimed at analysing the asset pricing implications of the existence of "systemic defaults". This framework is flexible and tractable enough to simultaneously replicate the price fluctuations of various far-out-of-the-money (disaster-exposed) credit and equity derivatives. According to our estimation results, market data imply that the default of a systemic entity is anticipated to be followed by a 4% decrease in consumption. The recessionary influence of systemic defaults implies that those financial instruments whose payoffs are exposed to such credit events carry substantial risk premiums.

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1 Introduction

Following the seminal contribution of Rietz (1988), many studies have shown that disaster risk helps solve many asset-pricing puzzles [see e.g. Barro (2006), Gabaix (2012), Gourio (2013)]. Disaster risk has notably be proven to successfully account for features of equity option markets [e.g. Du (2011), Wachter (2013), Tsai and Wachter (2015), Siriwardane (2016)], or credit derivatives [Collin-Dufresne et al. (2012) or Seo and Wachter (2016)].

In most of these asset-pricing studies, disasters are modelled as exogenous events causing dramatic increases in the default probabilities of bond issuers (or dramatic decreases in the asset values of firms). However, in some cases, this appears to be the default of a systemic entity *per se* that constitutes a disaster. Typically, the largest drop in the University of Michigan Consumer Sentiment index took place in September 2008, the month when Lehman Brothers went bankrupt. The existence of systemic entities is at the core of novel regulations on Systemically Important Financial Institutions (SIFIs) [International Monetary Fund (2010), Basel Committee on Banking Supervision (2013), Battiston et al. (2016)].

In this paper, we propose a general no-arbitrage asset-pricing framework where the defaults of some entities, called systemic entities, have economy-wide effects. The underlying credit risk model is that of Gouriéroux et al. (2014). This model features a finite number of homogeneous credit segments. The systemic entities are gathered in one or several segments. Systemic segments are such that the defaults of their constituent entities are contagious (within or between segments) and can affect macroeconomic or financial variables.[1] Because of contagion and feedback effects, a systemic default can be the source of default cascades, amplifying the costs of the original bankruptcy [Allen and Gale (2000), Stiglitz (2011)].[2]

While the model can be used in reduced form, we show that it can easily be incorporated in an endowment economy framework where the stochastic discount factor (s.d.f.) is derived from agents' preferences and from an exogenous consumption process.[3] Asset prices can then be computed once the relationship between their payoffs and the consumption process is specified. Crucially, we allow for an effect of systemic defaults on consumption.[4] Because bankruptcy cascades

[1]This paper falls in the category of "top-down" models, which focus on the default counting (or loss) processes [see e.g. Giesecke et al. (2011)], contrary to "bottom-up" approaches that consider the default processes of the individual firms as the the model primitives [see e.g. Lando (1998), Duffie and Singleton (1999) for defaultable bond pricing and Duffie and Gârleanu (2001) for the pricing of tranche products]. The "top-down" approach has been shown to satisfactorily capture the existence of default clusters [see Brigo et al. (2007), or Errais et al. (2010)]. In this approach, although default counts are modelled in the first place, one can recover underlying single-name default processes due to homogeneity assumptions.

[2]A recent theoretical study by Acemoglu et al. (2015) suggests that, due to non-linear mechanisms, interconnected systems – such as financial markets – are particularly fragile to large shocks.

[3]In our application, as in various recent contributions, our representative agent features Epstein and Zin (1989)'s preferences.

[4]Bruneau et al. (2012) find evidence of reciprocal links between the bankruptcy rate and real activity; they also

can be triggered by the defaults of a systemic entity, each systemic default is likely to eventually result in a sharp decline in consumption. In this context, financial instruments exposed to the default of systemic entities are expected to command substantial risk premiums, defined as those components of prices that would not exist if agents' were risk-neutral.

A key advantage of our framework is that it offers closed-from formulas to price a wide range of credit and equity market derivatives. Note that while the focus of our framework is on credit markets, it is easily extended to the equity market by positing a process for dividends.[5] Such a degree of tractability is for instance not present in the approach of Seo and Wachter (2016) or Collin-Dufresne et al. (2012), who have to resort to computer-demanding simulations so as to price synthetic Collateralised Debt Obligations (CDOs).[6] The tractability of our model facilitates its estimation. For instance, the model parameters can be estimated so as to optimise the fit of observed time series of a variety of derivatives prices.

Our empirical application demonstrates the ability of our model to capture, in a single and consistent framework, a substantial share of the joint fluctuations of stock and credit markets. The estimation is conducted on euro area data spanning the period from January 2006 to September 2017. We show that two factors allow to jointly account for the main fluctuations (i) of equity options written on the EURO STOXX 50 index, one of the main benchmarks of European equity markets, and (ii) of the iTraxx Europe main indices and associated CDOs. Our estimation procedure relies on the assumption that the 125 constituent entities of the iTraxx indices, that are the most liquid European investment grade credits, are systemic. For (i) and (ii), we put the emphasis on far-out-of-the-money (far-OTM) derivatives, that are financial instruments mostly exposed to (i) sharp decreases in the stock index or (ii) large number of defaults. We thereby test the ability of our model to reproduce market prices of instruments exposed to catastrophic events. As highlighted by Longstaff and Rajan (2008), CDO markets constitute an ideal laboratory to identify the joint distribution of default risk across firms – and therefore crash risks –, an information that cannot be inferred from the marginal distributions associated with single-name credit instruments. Let us stress that fitting tranche prices (CDOs) in an endowment economy model on a period that covers not only the pre-crisis period but also stress episodes distinguishes the present paper from previous studies [Seo and Wachter (2016), Christoffersen et al. (2017)].

highlight significant "second round effects" of shocks to the output gap on bankruptcies. This evidence is in line with the findings of Lown and Morgan (2006), who show that indicators of financial fragility, as measured by business failures, together with credit standards, have explanatory power for GDP growth.

[5] Azizpour et al. (2011), or Longstaff and Rajan (2008) consider the pricing of tranche products only. Christoffersen et al. (2017) consider the pricing of both stock and Credit Default Swaps (CDSs) in a joint model, but the model is changed to study synthetic Collateralised Debt Obligations (CDOs, i.e. tranche products). The connections between credit risk premiums and equity premiums is explored by Chen et al. (2009).

[6] A synthetic CDO is a tranche product written on a portfolio of credit default swaps (CDS). The portfolio can consist of an index of reference securities, such as the CDX or iTraxx indices [see e.g. Longstaff and Rajan (2008)].

Our findings point to the existence of substantial credit risk premiums in the credit derivatives written on systemic entities. In particular, our results suggest that about two thirds of 10-year CDS spreads written on systemic entities correspond to credit risk premiums. In other words, if agents were not risk-averse, these spreads would be three times lower. Besides, in line with previous studies [Azizpour et al. (2011), Giesecke and Kim (2011) or Brigo et al. (2009)], we find that an overwhelming share of the prices of the most senior tranches corresponds to risk premiums.

We also analyse the pricing of credit derivatives written on entities that are not systemic. Non-systemic entities are defined as entities whose default does not cause other entities' defaults and that have no macroeconomic impact. These non-systemic entities may however be exposed to systemic defaults – through contagion effects – and/or to other macroeconomic variables. We show that, for a fixed probability of default, the higher the exposure of these entities to systemic defaults, the higher the spreads of CDS written on these (non-systemic) entities.

As a by-product of our calibration exercise, we deduce estimates of the influence of systemic defaults on consumption (in the spirit of Backus et al. (2011)). Our calculations suggest that the default of a systemic entity is expected to be followed by a 4% decrease in consumption after two years, taking contagion effects into account. Let us provide some intuition for why this influence can be inferred from our estimation. Our equilibrium model provides some structure regarding credit risk premiums; that is, it determines how the size of risk premiums depends on the relationship between the payoffs of a given instrument and consumption. Conversely, through the lens of our model, indications regarding credit risk premiums convey information about the relationship between consumption and the instrument payoffs. For a CDS written on a systemic entity, the payoffs critically depend on systemic default. As a result, indications regarding the size of credit risk premiums included in a CDS spread written on a systemic entity shape the potential influence of a "systemic default" on consumption.

As in Bhansali et al. (2008), Santa-Clara and Yan (2010), or Giesecke and Kim (2011), we finally exploit our estimated model to derive systemic risk indicators. These indicators are defined as the probabilities of observing a certain number of systemic defaults over specific horizons.[7] The resulting systemic indicators reach their highest levels in late 2008, after the Lehman bankruptcy and in late 2011, when the European sovereign crisis was at its peak. In both instances, the probability to have at least 10 defaults of iTraxx constituents before two years was larger than 10%.

The remaining of this paper is organised as follows. Section 2 presents the general framework and provides derivatives pricing formulas. Section 3 shows how to exploit this general framework in an equilibrium macro-finance context and presents empirical results.

[7] Using prices of far-out-of-the-money put options to infer disaster probabilities dates back to Bates (1991). This approach has been applied recently by, among others, Bollerslev and Todorov (2011), Backus et al. (2011), Seo and Wachter (2016), Barro and Liao (2016), or Siriwardane (2016). For a discussion regarding the difficult measurement of systemic risk, see Hansen (2013).

2 Model

2.1 Overview and notations

We consider J homogeneous segments of defaultable entities. Each segment j is a segment of I_j entities with the same credit characteristics. The vector of number of entities per segment is $I = [I_1, \dots, I_J]'$ and the total number of entities is denoted by $I^* = \sum_j I_j$. If the entities are firms, the different segments can be defined by the industrial sector, the size, or the domicile country. Following the "top-down" approach, we model the default-count process instead of starting from the default intensities of the debtors ("bottom-up" approach).

We denote by $d_{j,i,t}$ the indicator of default of entity i belonging to segment j: $d_{j,i,t} = 1$, if entity i is in default at time t (or before) and $d_{j,i,t} = 0$, otherwise. The default indicators of all entities are gathered in vector $d_t = (d_{1,1,t}, \dots, d_{1,I_1,t}, \dots, d_{J,1,t}, \dots, d_{J,I_J,t})'$. We also denote by $N_{j,t}$ the number of segment-j entities in default at date t, i.e. $N_{j,t} = \Sigma_{i=1}^{I_j} d_{j,i,t}$, and by N_t the vector $N_t = (N_{1,t}, \dots, N_{J,t})'$. The total number of entities in default at date t (across segments) is denoted by $N_t^* = \sum_j N_{j,t}$.

We introduce a n_F-dimensional factor F_t that contributes to the default dependence. We denote by $\Omega_t = (\underline{F_t}, \underline{d_t})$, $\Omega_t^* = (\underline{F_{t+1}}, \underline{d_t}) = (F_{t+1}, \Omega_t)$ the information sets, where $\underline{F_t} = \{F_\tau, \tau \leq t\}$. These filtrations are nested in the following way: $\Omega_t \subset \Omega_t^* \subset \Omega_{t+1}$. The operator $\mathbb{E}_t$ denotes expectation conditional on the information available at time t, i.e. $\mathbb{E}_t(\bullet) = \mathbb{E}(\bullet|\Omega_t)$.

Finally, we introduce the vector $X_t = (F_t', N_t', N_{t-1}')'$.

2.2 Default-count process

Default-count dynamics is based on the following two assumptions:

Assumption 1. *Conditional on $\Omega_t^* = (F_{t+1}, \Omega_t)$, the default counts $n_{j,t} = \Delta N_{j,t}$, $j = 1, \dots, J$ are independent with Poisson distributions:*

$$n_{j,t+1}|\Omega_t^* \sim \mathscr{P}(\beta_j' F_{t+1} + c_j' n_t + \gamma_j). \tag{1}$$

Under Assumption 1, we have, in particular:

$$\mathbb{E}(n_{t+1}|\Omega_t^*) = \beta' F_{t+1} + c' n_t + \gamma \tag{2}$$

$$\mathbb{V}ar(n_{t+1}|\Omega_t^*) = \text{diag}(\beta' F_{t+1} + c' n_t + \gamma), \tag{3}$$

where $\beta = [\beta_1, \dots, \beta_J]$, $c = [c_1, \dots, c_J]$, $\gamma = [\gamma_1, \dots, \gamma_J]'$ and where $\text{diag}(w)$ denotes the diagonal matrix whose diagonal entries are the components of vector w.

Note that the Poisson distribution of eq. (1) has to be positive. This is in particular the case if all components of F_t, β, γ and c are non-negative.

Assumption 2. *The conditional Laplace transform of F_{t+1} given Ω_t is exponential affine in X_t:*

$$\mathbb{E}_t\left(\exp(w'F_{t+1})\right) = \exp\left(a(w)'F_t + b(w)'N_t + c(w)'N_{t-1} + d(w)\right), \text{ for any } w \in V, \tag{4}$$

where V denotes the set of vectors where this conditional Laplace transform exists. (It includes in particular the vectors with negative components.)

Proposition 1. *Under Assumptions 1 and 2, the log conditional Laplace transform of process (X_t), denoted by $\psi(v,X_t)$ and defined by:*

$$\mathbb{E}_t\left(\exp(v'X_{t+1})\right) = \exp(\psi(v,X_t)),$$

is affine in X_t. That is, we have:

$$\psi(v,X_t) = \psi_0(v) + \psi_1(v)'X_t,$$

where functions ψ_0 and ψ_1 are made explicit in Appendix A.

Proof. See Appendix A. □

Hence, under Assumptions 1 and 2, process (X_t) is affine. This has several important implications. In particular, conditional on Ω_t, first-order and second-order moments of future values X_t are given by affine functions of X_t. Appendix B gives an example of a process satisfying Assumption 2. In this example, the conditional distributions of the components of F_t are non-centered gamma distributions. Appendix B also provides formulas for the marginal and the first two conditional moments of X_t in this case. Such formula are useful when it comes to calibrate the model.

2.3 The stochastic discount factor (s.d.f.)

In order to define the pricing kernel from the historical distribution, we specify the form of the stochastic discount factor.

Assumption 3. *The stochastic discount factor is given by:*

$$M_{t,t+1} = \exp[-(\eta_0 + \eta_1'X_t) + \delta'X_{t+1} - \psi(\delta,X_t)]. \tag{5}$$

Vector δ is the vector of "prices of risk", which characterises the innovation of the s.d.f. [see e.g. Campbell (2000)]. Because, by definition, these innovations (from date $t-1$ to date t) do not depend on the entries of X_t corresponding to N_{t-1}, the s.d.f. does not depend on the components of δ that correspond to N_{t-1}.

Under Assumptions 1 to 3, we have $\mathbb{E}_t(M_{t,t+1}) = \exp[-(\eta_0 + \eta_1' X_t)]$. Therefore, under these assumptions, the riskfree short-term rate r_t between date t and $t+1$ is affine in X_t and given by:

$$r_t = \eta_0 + \eta_1' X_t. \tag{6}$$

Appendix C.1 demonstrates that, under Assumptions 1 to 3, there exist closed-form formula to compute, in particular, date-t prices of payoffs of the form $\exp(a'X_{t+h})$, $\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}$, $a'X_{t+h}$ or $a'X_{t+h}\mathbb{1}_{\{b'X_{t+h}<y\}}$ (settled on date $t+h$). As is shown in the next section, these formula are key building blocks to obtain pricing formula for specific financial instruments.

2.4 Pricing of Credit Default Swap

The *credit default swap* (CDS) is the most common credit derivative. It is an agreement between a protection buyer and a protection seller, whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit event, such as bankruptcy or failure to pay, of a reference entity. The contingent payment usually replicates the loss incurred by a creditor of the reference entity in the event of its default [See e.g. Duffie (1999)].

More specifically, a CDS works as follows: the *protection buyer* pays a regular (annual, semi-annual or quarterly) premium to the so-called *protection seller*. These payments end either after a given period of time (the maturity of the CDS) or at default of the reference entity i from segment j. In the case of the default of this debtor, the protection seller compensates the protection buyer for the loss the latter would incur upon default of the reference entity (assuming that the latter effectively holds a bond issued by the reference entity). The CDS spread, also called CDS premium, is the regular payment paid by the protection buyer (expressed in percentage of the notional and in annualized terms). Since, in our model, the segments of credit are homogeneous, the CDS spreads are the same for all entities belonging to the same segment. Let us denote by $S^{CDS}_{j,t,h}$ the maturity-h CDS spread of segment-j entities, by q the number of premium payments made per year and by RR the recovery rate.[8] At inception of the CDS contract, there is no cash-flow exchanged between both parties: Indeed, the CDS spread $S^{CDS}_{j,t,h}$ is determined so as to equalize the present discounted

[8]While the model is extensible to the case of stochastic recovery rates, we restrict our attention here to that of deterministic recovery rates as is common practice in pricing exotic credit derivatives.

values of the payments promised by each of them. If the maturity h is expressed in years, we have:[9]

$$\underbrace{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(1-RR)(d_{j,i,t+k}-d_{j,i,t+k-1})\right\}}_{\text{Protection leg}} = \underbrace{\mathbb{E}_t\left\{\frac{S^{CDS}_{j,t,h}}{q}\sum_{k=1}^{qh} M_{t,t+k}(1-d_{j,i,t+k})\right\}}_{\text{Premium leg}}. \quad (7)$$

By expanding the latter equality, it is clear that the CDS spread $S_{j,t,h}$ is easily derived if one can compute $\mathbb{E}_t(M_{t,t+k})$, $\mathbb{E}_t(M_{t,t+k}d_{j,i,t+k})$ and $\mathbb{E}_t(M_{t,t+k}d_{j,i,t+k-1})$ for all $k>0$. By symmetry arguments, assuming that $d_{j,i,t}=0$, we have:

$$\mathbb{E}_t(M_{t,t+k}d_{j,i,t+k}) = \mathbb{E}_t\left(M_{t,t+k}\frac{\overline{N}_{j,t+k}-\overline{N}_{j,t}}{I_j-\overline{N}_{j,t}}\right) = \frac{1}{I_j-\overline{N}_{j,t}}\left(\mathbb{E}_t(M_{t,t+k}\overline{N}_{j,t+k})-\overline{N}_{j,t}\mathbb{E}_t(M_{t,t+k})\right).$$

where $\overline{N}_{j,t}=\min(N_{j,t},I_j)$. While exact formula are available to compute these quantities, we will proceed under the assumption that the probability of having $N_{j,t}>I_j$ is so small that we have, in particular, $\mathbb{E}_t(M_{t,t+k}\overline{N}_{j,t+k})\approx\mathbb{E}_t(M_{t,t+k}N_{j,t+k})$. In this context, we obtain:

$$\mathbb{E}_t(M_{t,t+k}d_{j,i,t+k}) \approx \frac{1}{I_j-N_{j,t}}\left(\mathbb{E}_t(M_{t,t+k}N_{j,t+k})-N_{j,t}\mathbb{E}_t(M_{t,t+k})\right). \quad (8)$$

Similarly, it comes:

$$\mathbb{E}_t(M_{t,t+k}d_{j,i,t+k-1}) \approx \frac{1}{I_j-N_{j,t}}\left(\mathbb{E}_t(M_{t,t+k}N_{j,t+k-1})-N_{j,t}\mathbb{E}_t(M_{t,t+k})\right). \quad (9)$$

Therefore, as made explicit in Appendix C.2, approximated CDS spreads are easily obtained from the knowledge of $\mathbb{E}_t(M_{t,t+k})$, $\mathbb{E}_t(M_{t,t+k}N_{j,t+k})$ and $\mathbb{E}_t(M_{t,t+k}N_{j,t+k-1})$, whose computation stems from Corollary 1 (see Appendix C.1).

2.5 Pricing of credit indices

A *credit index* allows an investor to either buy or sell protection on a basket of reference entities. There are two main families of default swap indices, which serve as reference points for global

[9] This formula implicitly assumes that the model frequency matches the payment frequency, in the sense that spread payments take place at every period. This assumption can be relaxed, but this comes at the price of substantial notation complications. Alternatively, if one wants to use the presented formulas in a context where a time period is smaller than $1/q$ year, one can use these formulas to price spreads that would be consistent with higher-frequency payments and converts spreads accordingly afterwards (as is done when one prices (infrequent) coupon bonds using models where coupons are paid on a continuous basis).

CDS markets: Dow Jones CDX and iTraxx indices.[10] The U.S. Investment-Grade CDX main index and the iTraxx Europe main are comprised of 125 equally-weighted underlying credits (see Appendix E.1 for more details on the iTraxx index, which is used in our application).

In a credit index transaction, a protection seller agrees to pay all default losses in the index in return for a fixed periodic spread $S^{CI}_{t,h}/q$ paid on the total notional of obligors remaining in the index over a period of h years. Let us assume that the underlying portfolio consists of the equally-weighted component entities of a subset of m segments. Specifically, we assume that the segments entering the portfolio are those indexed by $\{j_1,\dots,j_m\}$. If we denote by $\tilde{I}$ the number of entities initially constituting the portfolio, we then have $\tilde{I}=\sum_{k=1}^{m} I_{j_k}$. Similarly, we denote by $\tilde{N}_t$ the number of entities the portfolio that have defaulted at pr before date t, i.e. $\tilde{N}_t=\sum_{k=1}^{m} N_{j_k,t}$. The index continues to exist after there is a credit event in any one of the reference entities. Should there be no credit event, the protection buyer pays a regular spread until maturity. Upon default of one the reference entity, the protection seller provides the buyer with the amount that the latter would have lost if she had held the index bond portfolio.[11] Then, following this default, the trade continues with the notional amount reduced by the weight of the defaulted credit.[12] The spread $S^{CI}_{t,h}$ is determined by equalizing the date-t values of the protection leg and of the premium leg, that is:

$$\underbrace{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(1-RR)\frac{\tilde{N}_{t+k}-\tilde{N}_{t+k-1}}{\tilde{I}}\right\}}_{\text{Protection leg}}=\underbrace{\frac{S^{CI}_{t,h}}{q}\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}\frac{\tilde{I}-\tilde{N}_{t+k}}{\tilde{I}}\right\}}_{\text{Premium leg}}. \tag{10}$$

Hence, credit index spreads result from the knowledge of $\mathbb{E}_t\left(M_{t,t+k}\tilde{N}_{t+k}\right)$ and $\mathbb{E}_t\left(M_{t,t+k}\tilde{N}_{t+k-1}\right)$, whose computation is addressed by Corollary 1.

2.6 Pricing of synthetic Collateralised Debt Obligations (CDOs)

Collateralised Debt Obligations (CDOs), or credit tranches, allow for the trading of credit-risk correlation or clustering. A credit tranche allows an investor to gain a specified exposure to the credit risk of the underlying portfolio, while in return receiving periodic coupon payments. Losses due to credit events in the underlying portfolio are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritized tranches (mezzanine tranches, followed by

[10] These indices are compiled, managed and owned by Markit, a financial services information company with a specific focus on credit derivatives pricing.

[11] For instance, for a \$100mm position in a 20-name index, with a recovery rate of 50%, the amount would be \$2.5mm ($=50\%\times 100/20$).

[12] In the example of the previous footnote, the new notional would be \$95mm; the number of reference entities in the index would be reduced to the remaining (non-defaulted) 19 entities.

senior tranches). The credit-tranche market consists of an actively traded segment and an illiquid "buy-and-hold" segment [Scheicher (2008)]. In the actively-traded segment, the underlying credit portfolio is based on the standardized portfolio of a credit index such as the iTraxx or the CDX index.[13]

The risk of a tranche is determined by so-called attachment and detachment points. The attachment point, denoted by a, determines the subordination of a tranche. The detachment point, denoted by b, determines the point beyond which the tranche has lost its complete notional. The equity tranche takes the first losses on the portfolio, from $a_1 = 0$ up to b_1. When the portfolio has accumulated large enough losses to exceed b_1 of notional, the next tranche, (a_2, b_2) (with $a_2 = b_1$), will incur losses from any additional defaults up to b_2 of losses, and so on. The difference between the attachment and detachment points is referred to as the thickness of tranche.

Let us detail the cash-flows induced by an (a,b) credit tranche. Consider a protection buyer and a protection seller who meet at date t. Their negotiation results in a spread $S^{TDS}_{t,h}(a,b)$, which is the quote associated with this credit tranche at date t. Let us denote by ℓ_t the ratio of cumulated loss, that is:

$$\ell_t = \frac{(1-RR)\tilde{N}_t}{\tilde{I}}.$$

From date $t+1$ to $t+h$, cash-flows are exchanged between the two parties, unless the cumulated losses ℓ_{t+k} (for $k = 1, \ldots, h$) have exceeded the detachment point b. Specifically, at date $t+k$, these cash-flows are the following:

- If cumulated losses ℓ_{t+k} have not reached the attachment point a: (i) there is no cash-flow paid by the protection seller and (ii) the protection buyer pays the full premium $S^{TDS}_{t,h}(a,b)/q$.
- If cumulated losses ℓ_{t+k} exceed the attachment point a but remain lower than the detachment point b: (i) the protection seller provides the protection buyer with an amount equal to the fraction of the tranche consumed by new losses between $t+k-1$ and $t+k$, that is $(\ell_{t+k} - \ell_{t+k-1})/(b-a)$, and (ii) the protection buyer pays a premium equal to the multiplication of the full premium $S^{TDS}_{t,h}(a,b)/q$ by the fraction of the tranche that has not been consumed at date $t+k$, that is $(b-\ell_{t+k})/(b-a)$.
- There are no subsequent cashflows when cumulated losses ℓ_{t+k} exceed the detachment point b.

[13] The less-actively-traded segment of the credit-tranche market consists of tailor-made tranches of Collateralised Debt Obligations (CDOs) in which various loans are bundled.

The spread $S^{TDS}_{t,h}(a,b)/q$ is such that the two legs have the same value at date t, that is:[14]

$$\underbrace{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}\frac{\ell_{t+k}-\ell_{t+k-1}}{b-a}\mathbb{1}_{\{a<\ell_{t+k}\le b\}}\right\}}_{\text{Protection leg}}$$
$$= \underbrace{U^{TDS}_{t,h}(a,b)+\mathbb{E}_t\left\{\frac{S^{TDS}_{t,h}(a,b)}{q}\sum_{k=1}^{qh} M_{t,t+k}\left(\mathbb{1}_{\{\ell_{t+k}\le a\}}+\frac{b-\ell_{t+k}}{b-a}\mathbb{1}_{\{a<\ell_{t+k}\le b\}}\right)\right\}}_{\text{Premium leg}}, \qquad (11)$$

where $U^{TDS}_{t,h}(a,b)$ is an upfront payment.[15] Credit tranches are either quoted in terms of spreads $S^{TDS}_{t,h}(a,b)$ or in terms of up-front payments $U^{TDS}_{t,h}(a,b)$. Typically, in the former case, the up-front payment is fixed, and vice versa.

Appendix C.3 shows that by expanding both sides of eq. (11), computing $S^{TDS}_{t,h}(a,b)$ – or, equivalently, $U^{TDS}_{t,h}(a,b)$ – amounts to calculating date-t prices of payoffs of the forms: (a) $\mathbb{1}_{\{\tilde{N}_{t+k}<z\}}$, (b) $\tilde{N}_{t+k}\mathbb{1}_{\{\tilde{N}_{t+k}<z\}}$ and (c) $\tilde{N}_{t+k-1}\mathbb{1}_{\{\tilde{N}_{t+k}<z\}}$ (the payoffs being settled at date $t+k$). The computation of such prices is addressed in Corollaries 2 and 3 (Appendix C.1).[16]

2.7 Stock returns and the pricing of equity derivatives

Let us denote by D_t the dividends paid by a stock whose date-t price is denoted by P_t. We have:

$$P_t = \sum_{h=1}^{\infty}\mathbb{E}_t(M_{t,t+h}D_{t+h}).$$

Assumption 4. *The log growth rate of dividends is affine in X_t. Formally:*

$$g_{d,t} = \mu_{d,0}+\mu'_{d,1}X_t. \qquad (12)$$

Proposition 2. *Under Assumptions 1 to 4, stock returns are approximately given by:*

$$r^s_{t+1} = \kappa_0 + A_0(\kappa_1-1)+\mu_{d,0}+(\kappa_1 A_1+\mu_{d,1})'X_{t+1}-A_1'X_t, \qquad (13)$$

[14] The price of the protection leg in eq. (11) is actually based on an approximation. The exact value of the protection leg is:

$$\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(\min(\ell_{t+k},b)-\max(\ell_{t+k-1},a))\mathbb{1}_{\{a<\ell_{t+k}\}}\mathbb{1}_{\{\ell_{t+k-1}\le b\}}\right\}.$$

[15] See e.g. O'Kane and Sen (2003), D'Amato and Gyntelberg (2005) or Morgan Stanley (2011) for an analysis of upfront versus running spread quoting conventions.

[16] Specifically, using the notations of Corollaries 2 and 3, these prices respectively correspond to (a) $g(0,\omega_0,z,h,X_t)$, (b) $\Gamma(\omega_0,\omega_0,z,h,X_t)$ and (c) $\Gamma(\omega_1,\omega_0,z,h,X_t)$, where $\omega_0=[\mathbf{0}_{1\times n_F},\mathbf{1}_{1\times J},\mathbf{0}_{1\times J}]'$ and $\omega_1=[\mathbf{0}_{1\times n_F},\mathbf{0}_{1\times J},\mathbf{1}_{1\times J}]'$.

where κ_0 and κ_1 are given in eq. (a.9) (Appendix C.4), where A_1 satisfies

$$\psi_1(\kappa_1 A_1 + \mu_{d,1} + \delta) = A_1 + \eta_1 + \psi_1(\delta),$$

and where

$$A_0 = \frac{-\kappa_0 - \mu_{d,0} + \eta_0 + \psi_0(\delta) - \psi_0(\kappa_1 A_1 + \mu_{d,1} + \delta)}{\kappa_1 - 1}.$$

Proof. See Appendix C.4. □

The payoffs of equity derivatives depend on P_t. The dynamics of P_t is completely defined by the ex-dividend return $\log(P_{t+1}/P_t)$, that we denote by r_t^*. This return is given by:

$$r_{t+1}^* = \log\left(\frac{P_{t+1}}{D_{t+1}} \times \frac{D_t}{P_t} \times \frac{D_{t+1}}{D_t}\right) = z_{t+1} - z_t + g_{d,t+1}, \tag{14}$$

where z_t denotes the log price-dividend ratio $\log(P_t/D_t)$, which can be approximated by $A_0 + A_1' X_t$ (see Appendix C.4). We therefore have, for any maturity $h \in \mathbb{N}$:

$$\begin{aligned} P_{t+h} &= P_t \exp\left(r_{t+1}^* + \cdots + r_{t+h}^*\right) && (15) \\ &= \exp\left(z_{t+h} - z_t + g_{d,t+1} + g_{d,t+2} + \cdots + g_{d,t+h}\right). && (16) \end{aligned}$$

Let us consider the price of a European put option of maturity h and strike K. This price is given by $\mathbb{E}_t\left(M_{t,t+h}(K - P_{t+h})\mathbb{1}_{\{K > P_{t+h}\}}\right)$. Using eq. (15), we obtain:

$$\begin{aligned} & \mathbb{E}_t\left(M_{t,t+h}(K - P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right) \\ = \; & K\mathbb{E}_t\left(M_{t,t+h}\mathbb{1}_{\{r_{t+1}^*+\cdots+r_{t+h}^* < \log(K) - \log P_t\}}\right) \\ & - P_t\mathbb{E}_t\left(M_{t,t+h}\exp(r_{t+1}^* + \cdots + r_{t+h}^*)\mathbb{1}_{\{r_{t+1}^*+\cdots+r_{t+h}^* < \log(K) - \log P_t\}}\right). \end{aligned} \tag{17}$$

Appendix C.5 provides details about the computation of the previous two conditional expectations when Assumptions 1 to 4 hold.

3 Empirical analysis

3.1 Additional assumptions

The previous section presents general pricing formula. In the present section, we show how these formula can be used in a more specific macro-finance context that we will bring to data.

Assumption 5. *The preferences of the representative agent are of the Epstein and Zin (1989)'s type, with a unit elasticity of intertemporal substitution (EIS). Specifically, the time-t utility of a consumption stream (C_t) is defined recursively by*

$$u_t = (1-\delta)c_t + \frac{\delta}{1-\gamma}\log\left(\mathbb{E}_t \exp\left[(1-\gamma)u_{t+1}\right]\right). \quad (18)$$

where c_t denotes the logarithm of the agent's consumption level C_t, δ denotes the time discount factor and γ is the risk aversion parameter.

Eq. (18) results from a first-order Taylor expansion around $\rho = 1$ of the general Epstein and Zin (1989)'s utility defined by

$$u_t = \frac{1}{1-\rho}\log\left((1-\delta)C_t^{1-\rho} + \delta\left(\mathbb{E}_t\left[\exp\{(1-\gamma)u_{t+1}\}\right]\right)^{\frac{1-\rho}{1-\gamma}}\right), \quad (19)$$

where ρ corresponds to the inverse of the EIS.

Using a unit EIS facilitates resolution.[17] Piazzesi and Schneider (2007) or Seo and Wachter (2016), among others, also work under this assumption of a unit EIS.

Under Assumption 5, we have:

$$\Delta u_t = \Delta c_t + \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\left\{\log\left(\mathbb{E}_t \exp\left[(1-\gamma)\Delta u_{t+1}\right]\right) - \log\left(\mathbb{E}_{t-1}\exp\left[(1-\gamma)\Delta u_t\right]\right)\right\}. \quad (20)$$

Assumption 6. *The log growth rate of consumption is given by:*

$$\Delta c_t = \mu_{c,0} + \mu_{c,1}' X_t. \quad (21)$$

Proposition 3. *Under Assumptions 1, 2, 3, 5 and 6, $\Delta u_t = \mu_{c,0} + \mu_{u,1}' X_t + (\mu_{c,1} - \mu_{u,1})' X_{t-1}$ satisfies eq. (20) for any $(X_t', X_{t-1}')'$ iff $\mu_{u,1}$ satisfies:*

$$\frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) + \frac{1}{1-\delta}(\mu_{c,1} - \mu_{u,1}) = 0, \quad (22)$$

where ψ_1 is introduced in Prop. 1 and made explicit in Appendix A.

Proof. See Appendix D.1. □

Proposition 4. *Under Assumptions 1, 2, 3, 5 and 6, we have:*

$$\log M_{t,t+1} = \mu_{m,0} + \mu_{m,1}' X_{t+1} + \mu_{m,2}' X_t,$$

[17] For other values of the EIS, approximate log-linearization can be resorted to (see e.g. Campbell (1993), Campbell (1996)).

with

$$\begin{cases} \mu_{m,0} & = & \log(\delta) - \mu_{c,0} - \psi_0((1-\gamma)\mu_{u,1}) \\ \mu_{m,1} & = & (1-\gamma)\mu_{u,1} - \mu_{c,1} \\ \mu_{m,2} & = & -\psi_1((1-\gamma)\mu_{u,1}). \end{cases}$$

Proof. See Appendix D.2. □

The previous proposition implies that, under Assumptions 1, 2, 3, 5 and 6, we also have:

$$r_t = -\log(\mathbb{E}_t(M_{t,t+1})) = -\mu_{m,0} - \psi_0(\mu_{m,1}) - [\psi_1(\mu_{m,1}) + \mu_{m,2}]' X_t.$$

That is:

$$\begin{cases} \eta_0 & = & -\mu_{m,0} - \psi_0(\mu_{m,1}) \\ \eta_1 & = & -[\psi_1(\mu_{m,1}) + \mu_{m,2}]. \end{cases}$$

3.2 A three-segment model

In this subsection, we consider a model with three segments of entities ($J = 3$). The first two segments of entities gather large firms supposed to be systemic. The only distinction between these first two segments is that the first contains the constituents of a priced credit index, which we will use to calibrate the model. Having a single segment of systemic entities would be restrictive because it would mean that our credit index has to cover all systemic entities in our economy, which may not be true. The third segment gathers non-systemic firms, that can be thought of as small firms.

The defaults of the firms of the first two segments are contagious. This is modelled by setting $c_{i,j} > 0$ for $i \in \{1,2\}$ and $j \in \{1,2\}$, where $c_{i,j}$, the i^{th} component of c_j (see eq. 1), determines the effect of the number of defaults in segment i on that of segment j. This means that we have both self- and mutual-excitation for the two segments of systemic entities. We assume further that there is contagion neither among nor from non-systemic firms, i.e. $c_{3,i} = 0$ for $i \in \{1,2,3\}$.

We assume that default probabilities are caused by two factors, namely $F_{1,t}$ and $F_{2,t}$.[18] More precisely, the number of defaults in the three segments, i.e. vector N_t, is Granger-caused by $F_{2,t}$, which, itself, is Granger-caused by $F_{1,t}$, the latter factor being autonomous. Without loss of generality, we impose $\mathbb{E}(F_{1,t}) = \mathbb{E}(F_{2,t}) = 1$. Appendix B.2 proposes a specification that is such that:

$$\begin{cases} F_{1,t} - 1 & = & \rho_1(F_{1,t-1} - 1) + \sigma_{1,t}\varepsilon_{1,t} \\ F_{2,t} - F_{1,t} & = & \rho_2(F_{2,t-1} - F_{1,t-1}) + \sigma_{2,t}\varepsilon_{2,t}, \end{cases} \tag{23}$$

[18] Collin-Dufresne et al. (2012) and Seo and Wachter (2016) also use two-factor models to price equity and credit derivatives, including tranche products. This allows to distinguish between long-run and short-run fluctuations of aggregate credit risk. Typically, while we had a relatively short-lived peak in (various) credit spreads in late 2008 - early 2009, the average level of spreads has remained higher several years afterwards (see e.g. Figure 2).

where $\varepsilon_t = [\varepsilon_{1,t}, \varepsilon_{2,t}]'$ is a martingale difference sequence with unit-variance components and where $[\sigma^2_{1,t}, \sigma^2_{2,t}]'$ is affine in $[F_{1,t-1}, F_{2,t-1}]'$ (see Appendix B.2). Importantly, this dynamics is built in such a way that $F_{1,t}$ and $F_{2,t}$ are non-negative.[19]

Intuitively, if $F_{1,t}$ is more persistent than $F_{2,t} - F_{1,t}$, i.e. if $0 < \rho_2 < \rho_1 < 1$, then $F_{1,t}$ can be seen as the low-frequency component of $F_{2,t}$. The residual component $F_{2,t} - F_{1,t}$, which has a marginal expectation of zero, can then be interpreted as the higher-frequency component of $F_{2,t}$.

Let us turn to the conditional distribution of n_t (Assumption 1). For any segment j, we assume that:

$$n_{j,t+1}|\Omega^*_t \quad \sim \quad \mathscr{P}(\beta_{j,2}F_{2,t+1} + c_{j,1}n_{1,t} + c_{j,2}n_{2,t}).$$

Still assuming symmetry between Segments 1 and 2, we impose $c_{1,1} = c_{1,2} = c_{2,1} = c_{2,2}$, as well as $c_{1,3} = c_{2,3}$. According to the latter equalities, the default of any systemic entity, from Segment 1 or 2, has the same influence on the default probabilities of any other systemic entities, be it of Segment 1 or 2. The former equality states that the defaults of systemic entities, be they of Segment 1 or 2, have the same influence on the default probabilities of third segment entities.

Let us turn to the specification of the consumption growth process (Assumption 6). We assume that systemic defaults have a negative impact on consumption growth. To have this, a possibility is to make Δc_t directly depend on the number of systemic defaults, i.e. $n_{1,t} + n_{2,t}$. However, this would have the nonrealistic implication that all systemic defaults have exactly the same effect on consumption growth. Therefore, we extend factor F_t with components $Z_{j,t} \equiv F_{2+j,t}$, $j \in \{1,2\}$ whose conditional distribution is:

$$Z_{j,t}|\Omega_t \sim \gamma_0(\beta_z n_{j,t-1}, \mu_z), \tag{24}$$

where the γ_0 distribution, introduced by Monfort et al. (2017), is a distribution featuring a Dirac mass at zero. In this context, if $Z_{j,t}$ is distributed as in eq. (24), then the probability that $Z_{j,t} = 0$, conditionally on Ω_{t-1}, is $\exp(-\beta_z n_{j,t-1})$. In particular, we have $Z_{j,t} = 0$ as long as there has been no systemic defaults in the previous periods. For identification reasons, the scale parameter μ_z is set to $1/(\beta_z \mathbb{E}(n_{1,t}))$, so as to have $\mathbb{E}(Z_{j,t}) = 1$.

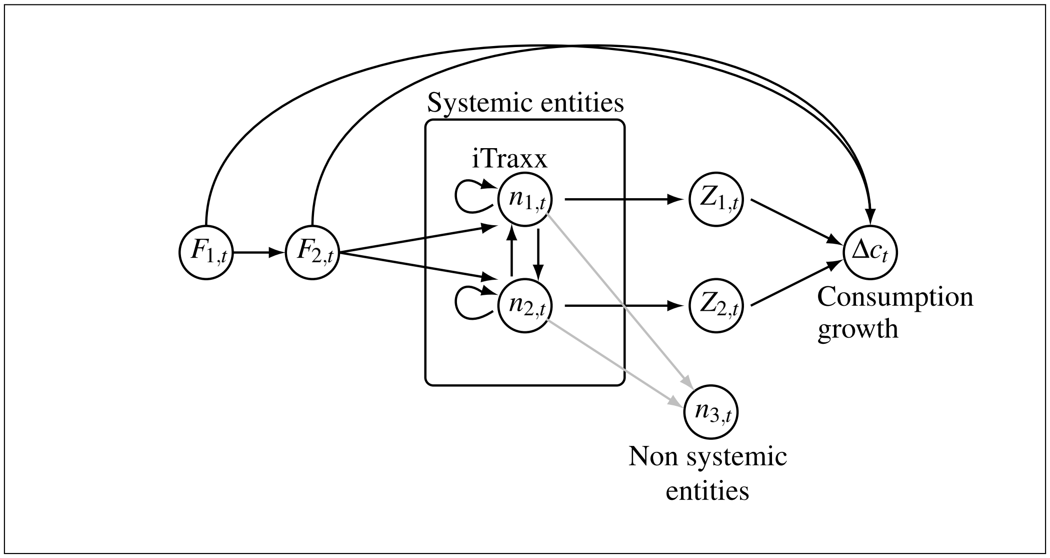
With these notations, eq. (21) is rewritten as:

$$\Delta c_t = \mu_{c,0} + \mu^F_{c,1} F_{1,t} + \mu^F_{c,2} F_{2,t} + \mu^{z\prime}_c Z_t. \tag{25}$$

Figure 1 provides a graphical representation of the resulting causality scheme.

[19] To obtain this, we employ auto-regressive gamma processes [see Gouriéroux and Jasiak (2006) or Monfort et al. (2017)].

Figure 1: Causality scheme



This figure provides a graphical representation of the causality scheme underlying the model. Arrows represent Granger-causality relationships.

As in Bansal and Yaron (2004), we finally assume that, up to a multiplicative factor χ, the sensitivity of the dividend growth rate to the state factors is the same as the one of the consumption growth rate. Accordingly, eq. (12) can be rewritten:

$$g_{d,t} = \mu_{d,0} + \chi \mu_{c,1}' X_t, \tag{26}$$

where $\mu_{d,0}$ is such that $\mathbb{E}(g_{d,t}) = \mathbb{E}(\Delta c_t)$. The parameter χ, also called leverage factor, is expected to be larger than one to reflect the fact that the growth rate of dividend is more volatile than the growth rate of consumption.

3.3 Model parameterization

3.3.1 Overview of the approach

To parameterize the model, we proceed as follows. First, some parameters are taken from the literature or are based on some priors we have concerning some processes. In a second step, the remaining parameters are estimated so as to minimize a loss function reflecting two types of targets: (i) there are population moments we would like our model to reproduce (*static targets* hereinafter) and (ii) we would like our model to capture the fluctuations of observed market prices (*dynamic targets*). Let us stress that we can implement such an approach because our model delivers closed-form solutions for population moments and asset prices. Indeed, if this was not the case, the time necessary to compute the loss function would be substantial, making a numerical optimization infeasible.

3.3.2 First step of the approach (pure calibration)

A critical part of the calibration regards the so-called preference parameters. Following Seo and Wachter (2016), we set the risk aversion parameter γ to 3 and the annualized rate of time preference to 1.2%. Because our model is at the bi-monthly frequency, this rate of time preference translates into $\delta = (1-1.2\%)^{1/6}$. As mentioned above, we consider a unit elasticity of intertemporal substitution ($\rho = 1$, see Assumption 5). The left panel of Table 1 displays additional calibrated parameters. In particular, consistently with the expected interpretation of $F_{1,t}$ and $F_{2,t}$ – the former being the low-frequency component of the latter – we set $\rho_1 = 0.98$ and $\rho_2 = 0.9$.[20] Another calibrated moment is the population expectation of consumption growth, that we set to 1.5% (annualized). We take a recovery rate RR of 50%, consistent with standard industry practice.

3.3.3 Second step of the approach (estimated parameters)

The second step of our approach consists in estimating the remaining parameters of the model so as to (approximately) fit our static and dynamic targets.

Data. We consider data covering the period from January 2006 to September 2017 at the bi-monthly frequency. We use credit index spreads and prices of tranches associated with the iTraxx Europe main index, whose constituents are 125 large European firms whose credit default swaps are actively traded (see Appendix E.1). As will be apparent in the result section, we consider several maturities and tranches, i.e. pairs of attachment/detachment points. Our financial data also include prices of far-out-of-the-money (far-OTM) equity options written on the EURO STOXX 50, which is one of the most important benchmarks of European equity markets. More precisely, we consider put options delivering strictly positive payoffs if the EURO STOXX 50 decreases by 30% (see Appendix E.2).

Static targets. First, we want the average default rate of the systemic entities to be of 0.3% per year. This is consistent with historical data on investment-grade entities compiled by Moody's.[21] Second, our target for the average short-term real risk-free rate is of 2% per annum, as in Christoffersen et al. (2017). Third, we define targets for the average level of CDS indices of two

[20] We also set $\mu_1 = 1-\rho_1$, which implies a unit variance for $F_{1,t}$ (see eq. a.4).

[21] More precisely, this corresponds to the average cumulative issuer-weighted global default rates for Baa-rated firms on the period 1920-2016 [see Moody's (2017), Exhibit 32]. In March 2016, the median rating for the iTraxx index (series 25) is BBB+ at S&P [Société Générale (2016)].

maturities: 3 and 10 years. We set these targets to 40 and 80 basis points, lower than their sample averages on 2006-2017 (about 60 and 110 basis points, respectively). We retain these (lower) levels so as to compensate for the fact that, in our estimation sample, the share of stress periods is high compared to a more standard period.

Dynamic targets. Our dynamic targets are the observations of (i) the iTraxx Europe main index, (ii) the associated tranche prices and (iii) the implied volatilities of the out-of-the-money equity options (see Appendix E).

Loss function. Let us denote by θ the vector of parameters to be estimated. The estimated vector of parameters is chosen so as to minimize the loss function $\mathscr{L}(\theta)$ defined by:

$$\mathscr{L}(\theta) = \sum_{i=1}^{4} \omega_i \left(m_i(\theta) - m_i^*\right)^2 + \sum_{i=1}^{k} \omega_{4+i} RMSE_i(\theta), \tag{27}$$

where the m_i^*'s and the $m_i(\theta)$'s are our static targets and their model counterparts, respectively, and where the $RMSE_i(\theta)$'s are the root mean squared errors stemming from our model.[22] The ω_i's are weights that have been chosen so as to obtain a balanced fit of our targets.

Factor estimates. In order to compute time series of model-implied prices – which are needed to compute the $RMSE_i(\theta)$'s – we need estimates of X_t. In the euro area, there has been no default of systemic entities over the period into consideration (2006–2017).[23] Accordingly, we have $n_{i,t} = Z_{i,t} = 0$ for $i \in \{1,2\}$ and all dates t in our sample. Therefore, estimating X_t amounts to estimating $F_{1,t}$ and $F_{2,t}$. To do so, we resort to the so-called inversion technique [see Chen and Scott (1993) or Ang and Piazzesi (2003)]. This technique consists in selecting observed prices that are assumed to be perfectly matched by the model. If the number of perfectly-matched series is the same as the number of factors driving these prices, then the latter can be backed out by inverting the pricing formulas. In our case, we get estimates of $F_{1,t}$ and $F_{2,t}$ by assuming that the 3-year and

[22] Because we consider observations (i) of CDS indices of 4 maturities, (ii) of tranche prices for 4 maturities and 5 attachment-detachment pairs for each maturity and (iii) of equity put prices for 2 maturities, we have $k = 26$ time series to fit (see Appendix E).

[23] On October 22 2009, the CDS contracts written on the French electronics firm Thomson were triggered. This entity was included among the iTraxx constituents. However, we do not consider this credit event to be a systemic event. Indeed, this credit event was not a default by the firm but a restructuring of its debt. In the U.S., following the so-called “Big Bang” changes in practices on credit events (April 8 2009) restructuring was excluded from the list of credit event triggering American CDS [see Coudert and Gex (2010)]. The recovery rate was determined through auctions; for the shortest maturity (2.5 years), the recovery rate was of 96.26%. This event had no noticeable repercussions on the credit market.

10-year iTraxx indices are modelled without pricing errors.[24,25]

3.4 Results

3.4.1 Model fit

Table 1 shows shows calibrated and estimated parameters. It notably appears that $c_{i,j}$ parameters ($i,j \in \{1,2\}$) are equal to 0.35, suggestive of a substantial level of contagion. Indeed, it implies for instance that an additional default by one systemic firm on date t leads to an increase in the expected number of systemic default on date $t+1$ by 0.70 (2×0.35) on date $t+1$. That kind of implication of a systemic default will be studied more extensively, by means of impulse response functions, in Subsection 3.4.2. We estimate a leverage factor χ of 2.16 (eq. 26), which is commensurate to the values used by Bansal and Yaron (2004) and Wachter (2013) (3 and 2.6, respectively).

Table 2 documents the fit resulting from our estimation approach. Panel (a) compares our static targets to their model-implied counterparts. This panel also reports a few additional resulting features of our model. It indicates for instance that the average excess return for our stock index is of 2.24% and that the maximum Sharpe ratio [Hansen and Jagannathan (1991)] has a reasonable value of 44%.[26] Panels (b), (c) and (d) of Table 2 compare the sample averages of observed financial data to their model-implied counterparts (i.e. the averages of the model-implied prices based on the estimated values of the state factors X_t, see Subection 3.3.3).

The fit of our dynamic targets is illustrated by Figures 2 to 5. Figure 2 illustrates the fit of the iTraxx indices of different maturities. Note that the fits of the 3-year and 10-year swaps are good by construction.[27] Figure 3 displays the model-implied growth rates of the stock index (r_t^*, see eq. 14) and compares them with the data. Recall that the extraction of our two risk factors is based on credit indices only (see the end of Subsection 3.3.3). Therefore, the fact that we have a relatively high correlation (60%) between model-implied and observed stock returns suggests that our overall approach is successful at capturing joint fluctuations of stock and credit markets. This is further illustrated by Figure 4, which compares observed and model-based implied volatilities

[24] In this case, inverting the formula is not trivial because the relationship between CDS indices and the $F_{i,t}$'s is not linear. However, a simple Gauss-Newton approach converges in a few iterations to the $F_{i,t}$'s yielding to a perfect fit of these two iTraxx quotes. To save computation time, a vectorial approach is implemented so as to estimate the $[F_{1,t}, F_{2,t}]$'s (for the different dates t) in a simultaneous way.

[25] Note that the inversion technique does not guarantee that the resulting $F_{i,t}$'s are non-negative (as they should do). As a result, we proceed as follows: after a preliminary estimation of the $F_{i,t}$'s, we replace negative solutions by zero. This implies that for those dates where the preliminary estimates of the $F_{i,t}$'s were negative, the fit is not perfect.

[26] This value is evaluated at the average values of the state vector X_t. The importance of Sharpe ratios to match empirical regularities across markets is highlighted by Chen et al. (2009). Appendix A.2 details the computation of the maximum Sharpe ratio in our context.

[27] The fit is actually perfect as long as the values of the factors $F_{1,t}$ and $F_{2,t}$ stemming from the inversion technique are non-negative. See Footnote 25.

of far-OTM put options. Figure 5 finally shows that the estimated model is able to reproduce the main fluctuations of the prices of iTraxx tranches for the three maturities and the five pairs of attachment/detachment points into consideration.

Taking into account the fact that we use a single model – and a single set of estimated state vectors – to simultaneously fit various market prices, the overall fit of our model is satisfactory. In particular, although we consider a longer period than Seo and Wachter (2016) and an additional maturity for tranche prices, our fit is comparable to theirs.[28]

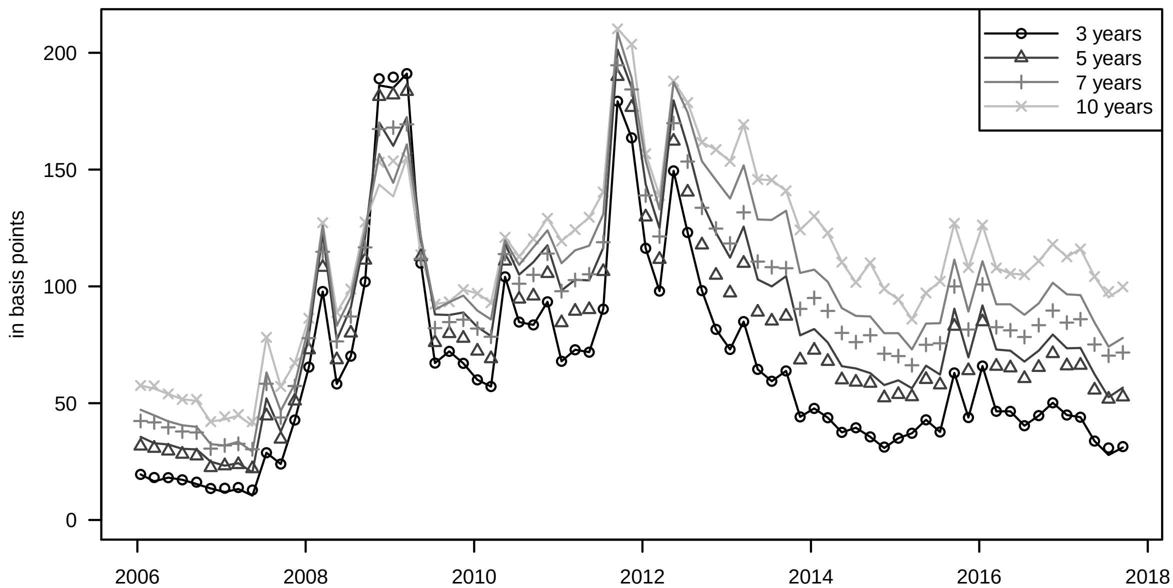
Table 1: Model parameterization

Panel (a) – Calibrated parameters			Panel (b) – Estimated parameters			
γ		3	$c_{i,j}$	$i,j \in \{1,2\}$		0.35
δ		0.997				
EIS		1.00	$\beta_{2,i}$	$i,j \in \{1,2,3\}$	$(\times 10^2)$	2.01
			β_z			0.13
ρ_1		0.98				
ρ_2		0.90	μ_c^z		$(\times 10^4)$	-8.26
			μ_2		$(\times 10^2)$	16.31
$\mathbb{V}ar(F_{1,t})$		1.00				
			χ			2.16
$\mathbb{E}(\Delta c_t)$	$(\times 6)$	1.50%				
$\mathbb{E}(g_{d,t})$	$(\times 6)$	1.50%	$\mu_{c,1}^F$		$(\times 10^5)$	-0.01
			$\mu_{c,2}^F$		$(\times 10^5)$	0.00

This table presents the model parameterization. $\mathbb{E}(\Delta c_t)$ is multiplied by 6 so as to be expressed in annualized terms. The parameterization is such that $\mathbb{E}(F_{1,t}) = \mathbb{E}(F_{2,t}) = 1$ (see Appendix B.2). The specification of the consumption growth rate is given by eq. (25). The specification of the dividend growth rate is given by eq. (26). Panel (a) reports parameters that are calibrated.Panel (b) reports those parameters that are estimated by minimzing the loss function $\mathscr{L}(\theta)$ given in eq. (27).

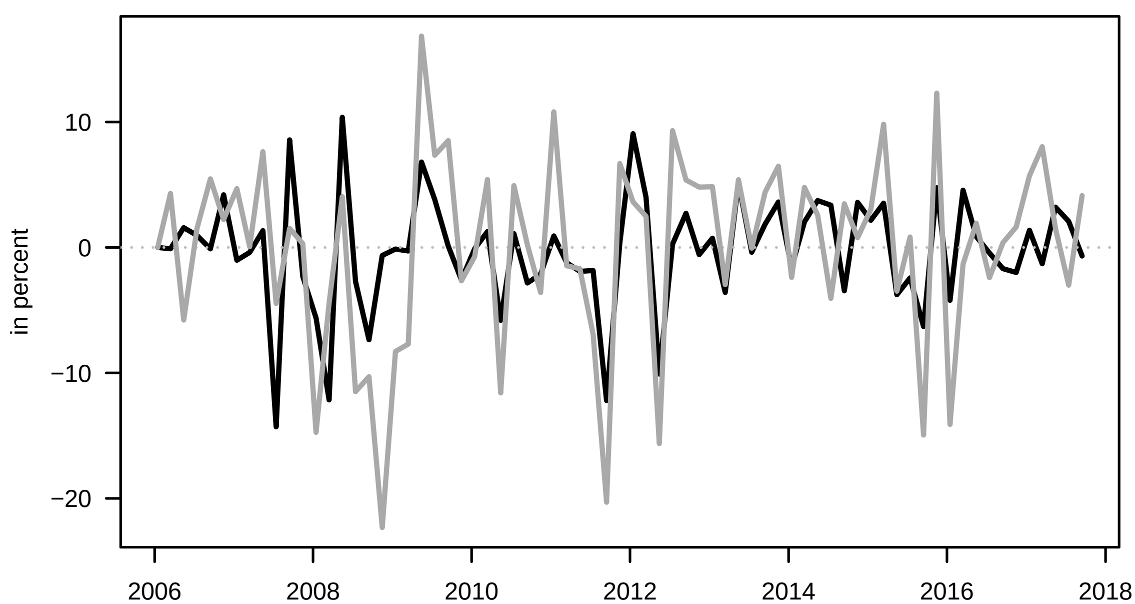
[28] Seo and Wachter (2016) also consider a two-factor model. Using a three-factor model, Christoffersen et al. (2017) obtain a better fit. However, they have an additional factor and consider a single maturity in their estimation.

Figure 2: Fit of iTraxx credit indices



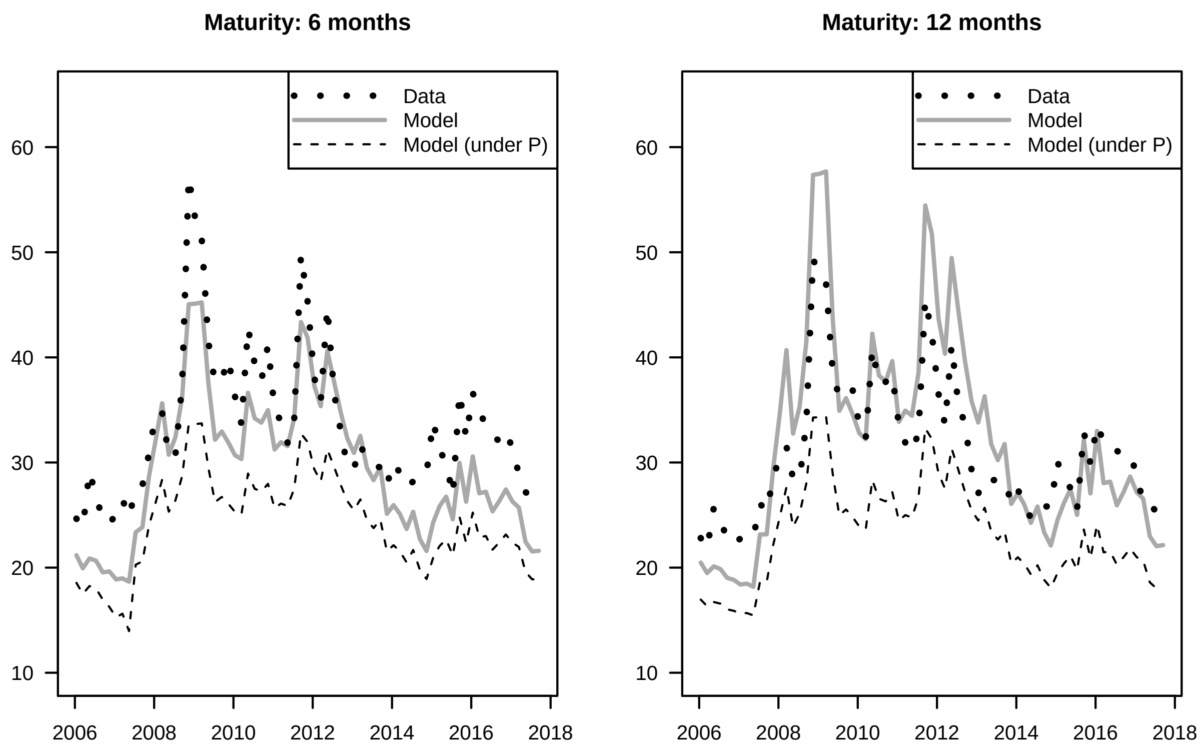
This figure displays iTraxx Europe main indices (solid lines) and their model-implied counterparts (symbols). The data cover the period from January 2006 to September 2017 at the bi-monthly frequency.

Figure 3: Observed versus fitted stock returns



This figure displays model-implied stock returns. More specifically, the black line shows r_t^* (see eq. 14). The grey line shows the returns of the EURO STOXX 50 index. The correlation between the two series is about 60%.

Figure 4: Fit of stock options



This figure displays implied volatilities of put options written on the EURO STOXX 50 index (black dots) and their model-implied counterparts (grey lines). The dashed black lines represent (model-based) implied volatilities that would prevail if agents were not risk averse (i.e. they correspond to the implied volatilities under the physical, or $\mathbb{P}$, measure).

Table 2: Model outputs

Panel (a) Population moments	Model	Target
Avg. short-term risk-free rate	2.10%	2.00%
St. dev. short-term risk-free rate	0.64%	
Avg. equity excess return	2.24%	
Maximum Sharpe ratio (at $X_t = \bar{X}$, for a one-year investment)	44.0%	
Avg. one-year default rate	0.34%	0.30%
Avg. 3-year ITRAXX	33 b.p.	40 b.p.
Avg. 10-year ITRAXX	58 b.p.	80 b.p.

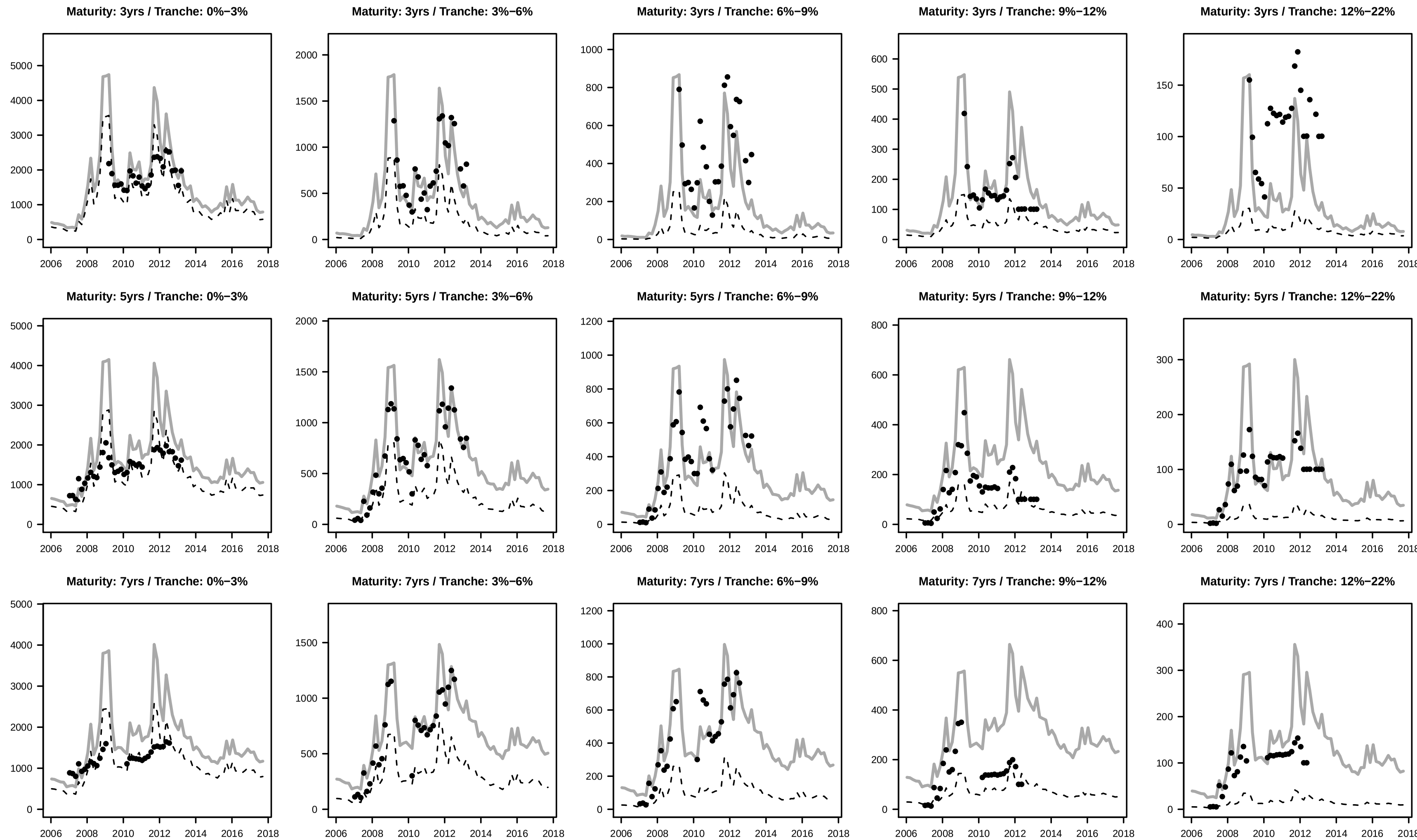
Panel (b) ITRAXX indices (sample averages, in b.p.)	Model	Data
3 years	65	65
5 years	80	88
7 years	92	101
10 years	112	112

Panel (c) ITRAXX tranches (sample averages, in b.p.)	Model	Data
3 years, Tranche: 0-3%	2369	1879
3 years, Tranche: 3-6%	738	772
3 years, Tranche: 6-9%	303	452
3 years, Tranche: 9-12%	220	160
3 years, Tranche: 12-22%	53	113
5 years, Tranche: 0-3%	2053	1444
5 years, Tranche: 3-6%	769	663
5 years, Tranche: 6-9%	416	421
5 years, Tranche: 9-12%	307	151
5 years, Tranche: 12-22%	120	91
7 years, Tranche: 0-3%	1975	1241
7 years, Tranche: 3-6%	773	672
7 years, Tranche: 6-9%	468	439
7 years, Tranche: 9-12%	336	146
7 years, Tranche: 12-22%	155	94

Panel (d) Implied Volatility (sample averages, in p.p.)	Model	Data
Maturity: 6 months	29%	33%
Maturity: 12 months	32%	30%

This table documents the fit of the model. $F_{1,t}$ and $F_{2,t}$ are backed out from the observations of the 3-year and 10-year iTraxx indices, assuming that these latter two prices are model without errors. If the resulting values of $F_{1,t}$ and $F_{2,t}$ are negative, then they are replaced by 0. This explains why, in spite of implementing the inversion technique, there may be a difference between model-implied and observed 3-year and 10-year iTraxx indices on some periods (resulting in differences in their respective sample averages). The reported maximum Sharpe ratio (see the online Appendix A.2 for its computation) is evaluated at the population mean of the state vector, i.e. for $X_t = \bar{X}$.

Figure 5: Fit of tranche prices, 3-years maturity



This figure displays model-implied iTraxx tranche prices (in grey). Black dots represent observed market prices. The dashed black lines correspond to counterfactual tranche prices that would be observed if agents were risk neutral (i.e. "under $\mathbb{P}$"). Therefore, the differences between the grey and black lines are credit risk premiums. All prices are expressed in basis points. For each date, maturity and tranche, we convert all quotes into an equivalent running spread with no upfront payment by using the risky duration approach [see e.g. O'Kane and Sen (2003), D'Amato and Gyntelberg (2005) or Morgan Stanley (2011)].

3.4.2 The effects of systemic defaults

This subsection examines the implication of our model regarding the effects of systemic defaults on stock and credit prices. To better understand the mechanisms at play, it is informative to first analyse the dynamic effects that a systemic default has on consumption. To do this, we rely on impulse response functions (IRFs) where the initial shock consists of an unexpected additional default by a systemic entity.[29] Figure 6 displays the results.

The left-hand panel of Figure 6 shows the dynamic responses of the number of systemic defaults following an unexpected systemic default on date $t = 0$. Because of contagion phenomena, the initial default increases the expected number of subsequent systemic defaults. More precisely, it appears that a systemic default triggers two additional systemic defaults in the subsequent two years, on average (one additional default in Segment 1 and another one in Segment 2). The middle panel shows, in black, the response of consumption following the systemic default. This response is gradual, going from 0 to -4% in the two years following the shock. The economic impact of a systemic default is therefore substantial.[30] Interestingly, in our model, a systemic default has not only an impact on conditional expectations, but also on conditional variances: upon arrival of a systemic default, we observe a jump of the volatility of consumption growth, i.e. a dramatic increase in economic uncertainty (right-hand plot of Figure 6).

The middle and right-hand panels of Figure 6 also display the responses of stock returns r_t^*. Following a systemic default, the conditional level and volatility of the stock index undergo the same influences as consumption does, except that the responses are expanded. This expansion essentially reflects the existence of the leverage factor χ (see eq. 26).

Let us now turn to the study of credit risk premiums, which we define as the differences between model-implied prices and those prices that would be observed if agents were not risk averse. The latter prices are computed by replacing the s.d.f. $M_{t,t+1}$ by $\exp(-r_t)$ in the pricing formulas. Such counterfactual prices are said to be computed under the physical, or $\mathbb{P}$, measure; standard model-implied prices are said to be computed under the risk-neutral, or $\mathbb{Q}$, measure.[31]

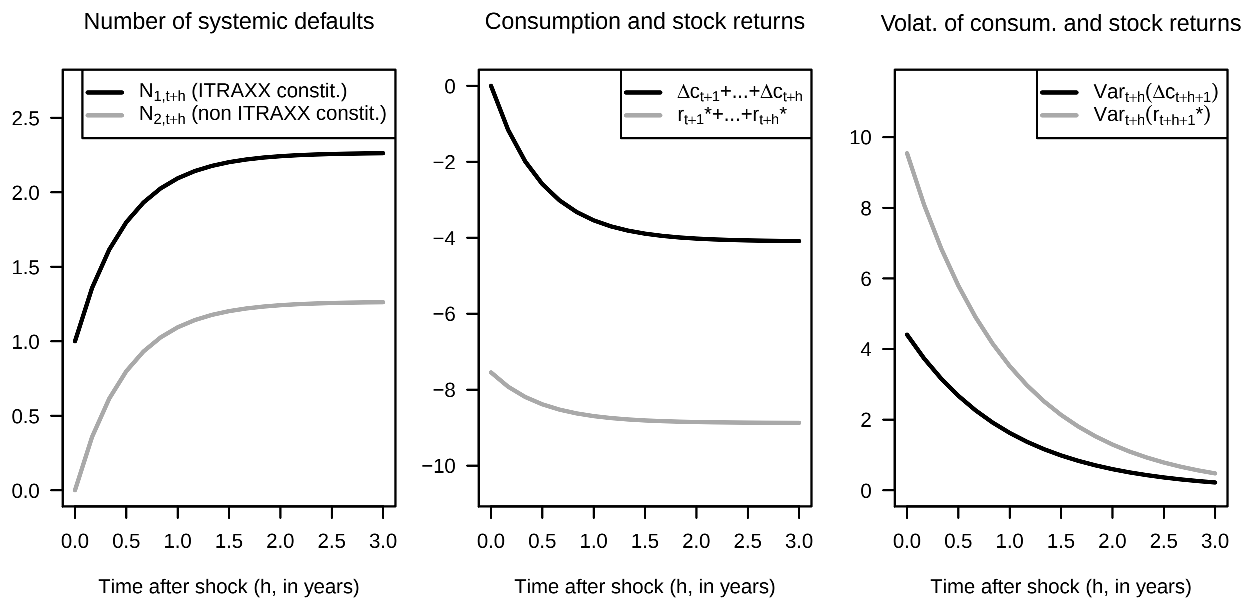
Figure 7 displays the $\mathbb{P}$ (grey) and $\mathbb{Q}$ (black) CDS spreads for two maturities, 5 and 10 years. The differences between the two types of spreads are credit risk premiums. The solid lines corre-

[29]Formally, for different variables Y_t (including consumption growth Δc_t), we consider $\mathbb{E}(Y_{t+h} - \mathbb{E}(Y_t)|n_{1,t} = \mathbb{E}(n_{1,t}) + 1)$ for different h. Note that, in our set-up, these IRFs are straightforward to compute as long as the variable Y_t into consideration is an affine function of the state vector X_t.

[30]For the sake of comparison, Laeven and Valencia (2012) find that a systemic banking crisis is, on average, followed by a 23% decrease in output, which would therefore correspond to about 6 defaults of systemic entities (assuming that consumption and GDP move in tandem).

[31]The price of an asset whose date-t payoff is P_t satisfies the Euler equation $P_t = \mathbb{E}_t(M_{t,t+1}P_{t+1})$. Hence, denoting by $\mathbb{Q}$ the measure defined through the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P} = \exp(r_t)M_{t,t+1}$, the Euler equation can be rewritten $P_t = \mathbb{E}_t^{\mathbb{Q}}(\exp(-r_t)P_{t+1})$. This price ("price under $\mathbb{Q}$") can be compared to the price that would be observed if agents were not risk averse ("price under $\mathbb{P}$"): $\mathbb{E}_t(\exp(-r_t)P_{t+1})$.

Figure 6: Responses to an unexpected default of a systemic entity



This figure displays impulse response functions (IRFs) associated with an additional default of a (iTraxx) systemic entity at date $t = 0$. That is, the initial shock is $n_{1,t=0} = \mathbb{E}(n_{1,t}) + 1$. The left-hand panel displays the reactions of the number of systemic defaults. There are two segments of systemic entities: Segment 1 (Segment 2) gathers those entities that are (that are not) constituents of the iTraxx index. The middle panel displays changes in expectations of future consumption and of future stock index. The right-hand panel shows the effect on the expectations of future conditional variances of consumption growth and of stock returns. To facilitate the reading, we plot the square roots of the expected conditional variance.

spond to spreads of CDS written on systemic entities. In late 2011, CDS premiums accounted for almost three quarters of the 10-year CDS spread. Such high risk premiums reflects the fact that the default of a systemic entity is a particularly bad state of the world, i.e. a state of high marginal utility: when it happens, agents dramatically revise their future consumption path downward (consistently with the IRF plotted on the middle panel of Figure 6). In the context of a CDS written on a systemic entity, the seller of protection therefore expects to face large losses in bad states of the world. As a result, she is willing to provide this protection only if the compensation is high enough, i.e. if the CDS spread is sufficiently above her expected loss, which translates into high credit risk premiums.

The dotted lines in Figure 7 correspond to $\mathbb{P}$ (grey) and $\mathbb{Q}$ (black) CDS spreads associated with non-systematic entities. Note that, at this stage, we have not discussed the parameterization of the number of non-systemic defaults ($n_{3,t}$). Because this number does not cause any other variable in the model (see Figure 1), it affects none of the prices we have considered until then. In particular, it was not necessary to parameterize the conditional distribution of $n_{3,t}$ to estimate the model. This means that we now are free to choose the exposure of non-systemic entities to the risk factors. The dotted lines are for instance obtained for the following exposures: $\beta_{2,3} = 4 \times \beta_{2,1}$ ($= x$, say) and $c_{1,3} = c_{2,3} = 0$ ($= y$, say). The former equation means that Segment-3 entities are four times more exposed to $F_{2,t}$ than systemic entities; the latter set of equalities means that these entities are not directly exposed to systemic defaults. These arbitrary exposures (x, y) have been chosen so that Segment-3 entities feature approximately the same average default probability than the systemic entities (Segments 1 and 2). However, Figure 7 shows that the spreads of CDS written on these entities are far lower than those for systemic entities. This figure also shows that the "$\mathbb{P}$ parts" of the CDS spreads are quite similar for the systemic entities and for Segment-3 entities. The last point was expected because $\mathbb{P}$ CDS spreads essentially reflect default probabilities and Segment-3 entities have, on average, approximately the same default probability as systemic entities. The reason why credit risk premiums are lower for Segment-3 entities is that the defaults of such entities tend to occur in *relatively* better states of the world than is the case for systemic entities. Intuitively, though defaults of Segment-3 entities tend to happen when $F_{2,t}$ is high, the decline in consumption may then remain subdued as long as such a high level of $F_{2,t}$ has not triggered (recessionary) defaults of systemic entities.

Again, the exposure (x, y) chosen for the Segment-3 entities was arbitrary. Naturally, another exposure (x, y) would have resulted in different dotted lines in Figure 7. In particular, we could have chosen $x = \beta_{2,3} < \beta_{2,1}$ and $y = c_{1,3} = c_{2,3} > c_{1,1} = c_{2,1}$ (say), still keeping the average default probability constant. In this case, compared to systemic entities, a larger fraction of Segment-3 entities would take place in particularly bad states of the world. Accordingly, we would then expect higher CDS spreads for this new type of entities than for the systemic ones. Note that they would however remain "non systemic" because their default would still not cause consumption growth or other defaults.

Figure 8 explores in a more systematic way the relationship between the exposures to the risk factors (x, y) on the one hand, and the $\mathbb{Q}$-$\mathbb{P}$ ratio on the other hand. (The the $\mathbb{Q}$-$\mathbb{P}$ ratio is the ratio between model-implied CDS spreads and the counterfactual $\mathbb{P}$ CDS spreads.) On Figure 8, we connect, with black lines, those pairs of exposures resulting in the same average $\mathbb{Q}$-$\mathbb{P}$ ratio, for the 10-year maturity. We also connect, with dashed grey lines, pairs of exposures resulting in the same average one-year probability of default. While the black square represents the Segment-3 entities we considered in Figure 7 (dotted lines), the circle indicates an entity that features the same exposures as our systemic entities. It appears that the average default probabilities of these two types of entities are close (between 0.3% and 0.4%) but their $\mathbb{Q}$-$\mathbb{P}$ ratios differ substantially (2.5 and 1.5). The figure also shows that, for each average probability of default, there exists a maximum $\mathbb{Q}$-$\mathbb{P}$ ratio. Typically, for a one-year probability of default of 0.4%, the maximum $\mathbb{Q}$-$\mathbb{P}$ ratio is about 2.9.

Credit risk premiums are also present in iTraxx tranche spreads. On Figure 5, these risk premiums are the differences between the grey lines and the dashed black lines: while the grey lines are the model-implied tranche prices, the dotted lines are their $\mathbb{P}$ counterparts, i.e. the (model-implied) prices that would prevail if agents were not risk averse. It appears that the more senior the tranche, the higher the relative importance of credit risk premiums. This is consistent with the fact that more senior tranches are more exposed to catastrophic events.

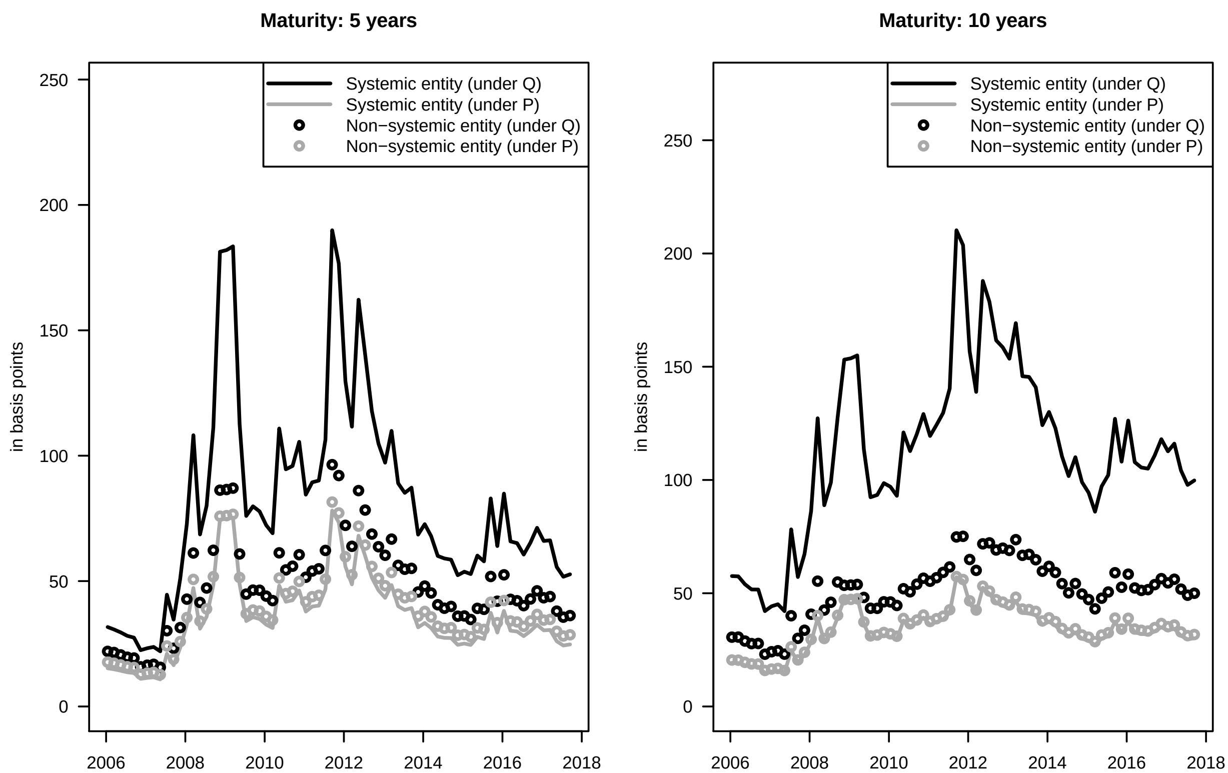
3.4.3 Measuring systemic risk

Our approach provides us with natural measures of systemic risk. Our indicators are defined as the probability to have at least q systemic defaults (say) at any horizon h.[32]

As an illustration, Figure 9 plots the probability to observe at least 10 defaults of iTraxx constituents in the next 12 months (grey line) and 24 months (black line). We also report vertical lines indicating significant dates of the financial crisis. This shows that our systemic indicators reached their maximum levels in late 2008, after the Lehman bankruptcy and in late 2011, when the European sovereign crisis was at its peak. In both instances, the probability to have at least 10 defaults of iTraxx constituents before two years was larger than 10%.

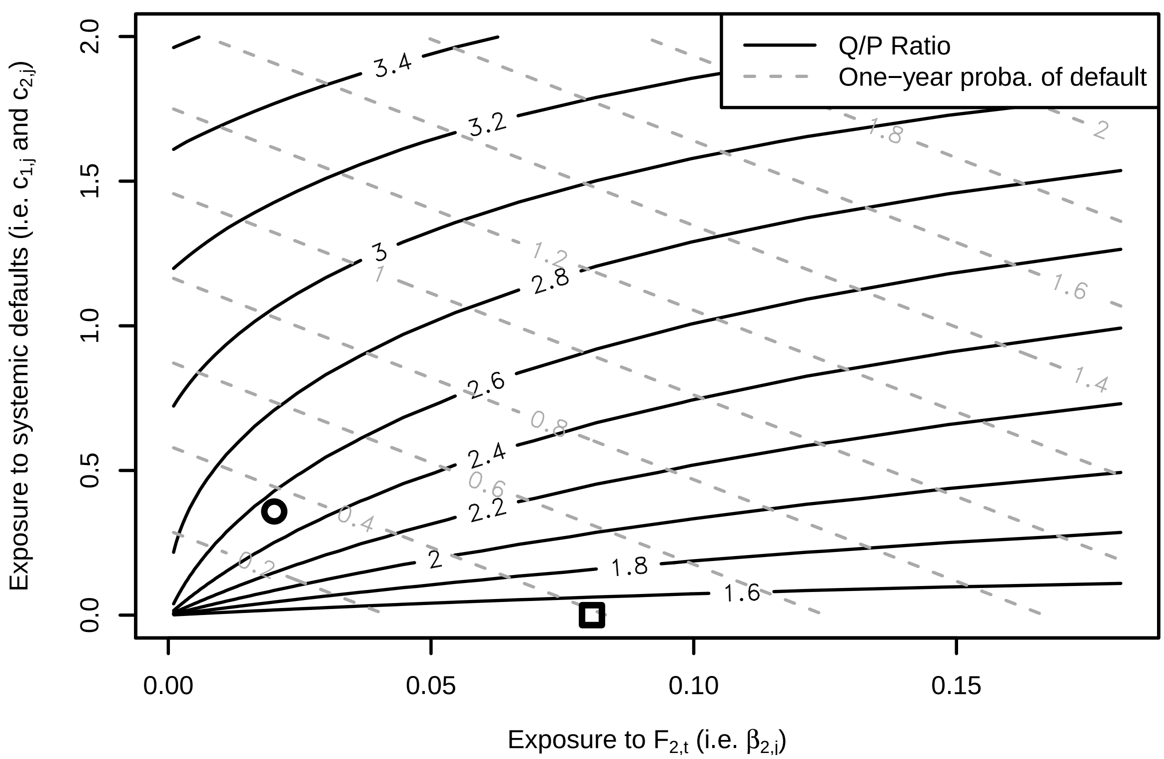
[32]Closed-form formula can be deduced from a straightforward adaptation of Corollary 2.

Figure 7: Credit risk premiums in iTraxx Europe main indices



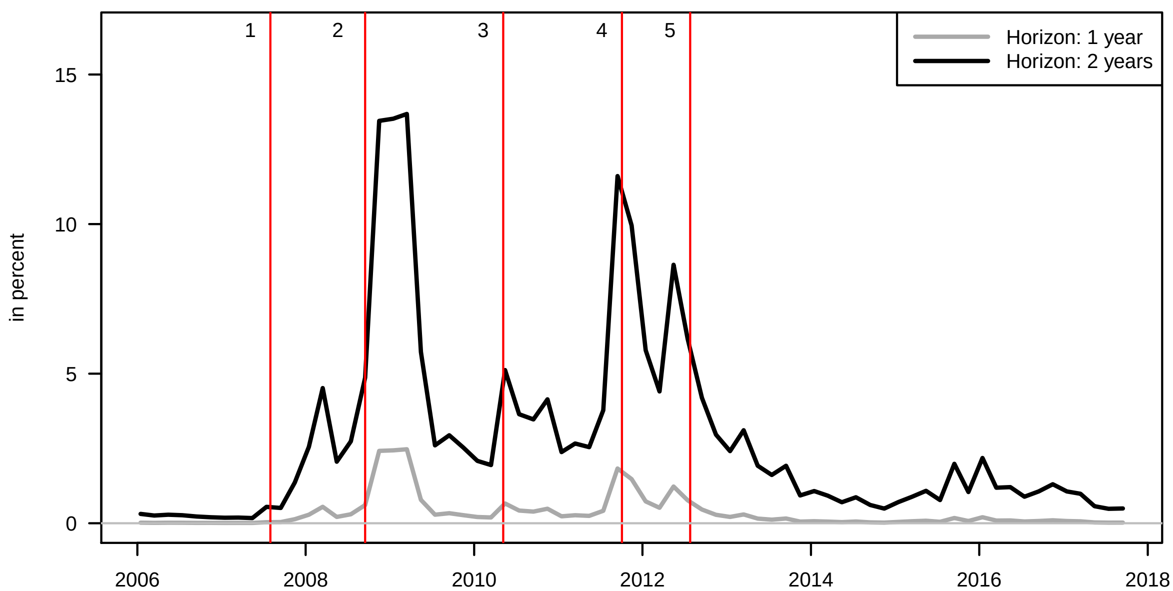
This figure illustrates the importance of credit risk premiums in iTraxx Europe main indices. The black solid line is the model-implied iTraxx index. The grey solid line is the (counterfactual) iTraxx index that would prevail if agents were not risk averse (said to be the iTraxx index "under the physical measure $\mathbb{P}$"). The difference between the black and grey solid lines reflects credit risk premiums. The dotted lines correspond to ($\mathbb{P}$ and $\mathbb{Q}$) CDS spreads associated with a firm from the third segment. See Subsection 3.4.2 for more details.

Figure 8: Relationship between exposures to risk factors and importance of credit risk premiums



This figure illustrates the influence of the exposure to the risk factors on the relative importance of risk premiums in CDS indices. The coordinates of each point correspond to the exposure of a given non-systemic entity to factor $F_{2,t}$ (abscissa) and to the number of systemic defaults, i.e. $n_{1,t} + n_{2,t}$ (ordinate). The black lines connect those pairs of exposures implying the same Q-P ratio, that is the ratio between the (model-implied) CDS spread and the counterfactual CDS spread that would be observed if agents were not risk averse. (The former is the one computed under the pricing, or risk-neutral, measure ($\mathbb{Q}$); the latter is computed under the physical measure ($\mathbb{P}$), hence the name "Q-P ratio".) We consider the 10-year maturity. The grey dashed lines connect pairs of exposures implying the same average probability of default. Figures reported in grey are probabilities of default expressed in annualized percentage points. The circle indicates a pair of exposures corresponding to the systemic entities. The square indicates the pair of exposures of those non-systemic entities whose CDS indices are displayed on Figure 7.

Figure 9: Probability that at least 10% of iTraxx constituents default in the next two years



This figure displays the (model-implied) probabilities that at least 10% of the iTraxx constituents – considered to be systemic entities – default in the coming 12 months (grey line) and 24 months (black line). The vertical bars correspond to important dates of the financial crisis (see Bruegel, http://bruegel.org/2015/09/euro-crisis/): (1) August 2007: European interbank markets seize-up; (2) 15 September 2008: Collapse of Lehman Brothers; (3) 7 May 2010: Emergency measures to safeguard financial stability; (4) October 2011: Spain and Italy are hit by a wave of rating downgrades by the three main rating agencies; (5) 26 July 2012: ECB President Mario Draghi says that the ECB will do "whatever it takes to preserve the euro".

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A Proof of Proposition 1

We have:

$$\begin{aligned}
& \mathbb{E}_t\left(\exp((v_A',v_B',v_C')X_{t+1})\right)\\
= & \mathbb{E}_t\left(\exp(v_A'F_{t+1}+v_B'N_{t+1}+v_C'N_t)\right)\\
= & \mathbb{E}_t\left(\mathbb{E}\left(\exp(v_A'F_{t+1}+v_B'N_{t+1}+v_C'N_t)\,\middle|\,\Omega_t,F_{t+1}\right)\right)\\
= & \mathbb{E}_t\left(\exp(v_A'F_{t+1}+(v_B+v_C)'N_t)\mathbb{E}\left(\exp(v_B'(N_{t+1}-N_t))\,\middle|\,\Omega_t^*\right)\right)\\
= & \mathbb{E}_t\left(\exp(v_A'F_{t+1}+(v_B+v_C)'N_t)\mathbb{E}\left(\exp\left[\sum_{j=1}^{J}v_{B,j}(N_{j,t+1}-N_{j,t})\right]\,\middle|\,\Omega_t^*\right)\right)\\
= & \mathbb{E}_t\left(\exp\left(v_A'F_{t+1}+(v_B+v_C)'N_t+\sum_{j=1}^{J}\left(\beta_j'F_{t+1}+c_j'(N_t-N_{t-1})+\gamma_j\right)(e^{v_{B,j}}-1)\right)\right)\\
& \text{(since the } n_{j,t+1}\text{s, are independent conditional on } \Omega_t^*)\\
= & \exp\left(\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)c_j+v_B+v_C\right\}'N_t-\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)c_j\right\}'N_{t-1}+\sum_{j=1}^{J}\gamma_j(e^{v_{B,j}}-1)\right)\times\\
& \mathbb{E}\left(\exp\left(\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)\beta_j+v_A\right\}'F_{t+1}\right)\,\middle|\,\Omega_t\right).
\end{aligned}$$

Therefore, we have:

$$\log\mathbb{E}\left(\exp((v'X_{t+1})\right)=\psi_1(v)'X_t+\psi_0(v),$$

with

$$\psi_0(v)=d(\sum_{j=1}^{J}(\exp(v_{B,j})-1)\beta_j+v_A)+\sum_{j=1}^{J}(\exp(v_{B,j})-1)\gamma_j,$$

and where $\psi_1(v)=(A(v)',B(v)',C(v)')'$, where:

$$\begin{cases}
A(v) & =a(\sum_{j=1}^{J}(\exp(v_{B,j})-1)\beta_j+v_A)\\
B(v) & =b(\sum_{j=1}^{J}(\exp(v_{B,j})-1)\beta_j+v_A)+\sum_{j=1}^{J}(\exp(v_{B,j})-1)c_j+v_B+v_C\\
C(v) & =c(\sum_{j=1}^{J}(\exp(v_{B,j})-1)\beta_j+v_A)-\sum_{j=1}^{J}(\exp(v_{B,j})-1)c_j,
\end{cases}$$

with $v=(v_A',v_B',v_C')'$, v_A being a n_F-dimensional vector and v_B and v_C being J-dimensional vectors.

B An example of conditional distribution of F_t

B.1 General case

Let us assume that, conditional on Ω_t, the different components of F_{t+1} are independent and drawn from non-centered Gamma distributions:[33]

$$F_{i,t+1}|\Omega_t \sim \gamma_{\nu_i}(\zeta_{i,0}+\zeta_{i,F}'F_t+\zeta_{i,n}'n_t,\mu_i),$$

where ν_i, $\zeta_{i,0}$ and μ_i are scalar, $\zeta_{i,F}$ is a n_F-dimensional vector and $\zeta_{i,n}$ is a J-dimensional vector.

In this case, eq. (4) is satisfied, with:

$$\begin{aligned} a(w) &= \zeta_F\left(\frac{w\odot\mu}{1-w\odot\mu}\right), \quad b(w)=\zeta_n\left(\frac{w\odot\mu}{1-w\odot\mu}\right), \quad c(w)=-b(w), \\ d(w) &= \zeta_0'\left(\frac{w\odot\mu}{1-w\odot\mu}\right)-\nu'\log(1-w\odot\mu), \end{aligned} \tag{a.1}$$

with $\zeta_F=[\zeta_{1,F},\dots,\zeta_{n_F,F}]$, $\zeta_n=[\zeta_{1,n},\dots,\zeta_{J,n}]$, $\zeta_0=[\zeta_{1,0},\dots,\zeta_{n_F,0}]'$, $\mu=[\mu_1,\dots,\mu_{n_F}]'$, $\nu=[\nu_1,\dots,\nu_{n_F}]'$, where $\odot$ is the element-by-element (Hadamard) product and where, by abuse of notations, the log and division operator are applied element-by-element wise.

We have that:

$$\begin{aligned} \mathbb{E}(F_{t+1}|\Omega_t) &= \mu\odot(\zeta_0+\nu)+\mu\odot(\zeta_F'F_t+\zeta_n'n_t) \qquad \text{(a.2)} \\ &=: \mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t \\ \mathbb{V}ar(F_{t+1}|\Omega_t) &= \text{diag}\left[\mu\odot\mu\odot(2\zeta_0+\nu)+2(\{\mu\odot\mu\}\mathbf{1}')\odot(\zeta_F'F_t+\zeta_n'n_t)\right] \qquad \text{(a.3)} \\ &=: \text{diag}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right), \end{aligned}$$

where $\mathbf{1}$ is a n_F-dimensional vector of ones.

An online appendix (A.1) provides formulas to compute conditional and unconditional first two moments of $[F_t',n_t']'$ in the present case (and when Assumption 1 is satisfied).

[33]The random variable W is drawn from a non-centered Gamma distribution $\gamma_\nu(\varphi,\mu)$, iif there exists a $\mathscr{P}(\varphi)$-distributed variable Z such that $W|Z\sim\gamma(\nu+Z,\mu)$ where Z and μ are, respectively, the shape and scale parameters of the Gamma distribution [see e.g. Gouriéroux and Jasiak (2006)]. When $Z=0$ and $\nu=0$, then $W=0$. When $\nu=0$, this distribution is called Gamma$_0$ distribution; this case is introduced and studied by Monfort et al. (2017).

B.2 A specific case

Consider the specific case of Appendix B.1 where F_t is of dimension two and where $\zeta_n = 0$. We have that $F_{1,t}$ Granger-causes $F_{2,t}$ but that the reverse is not true if and only if ζ_F is of the form:

$$\zeta_F = \begin{bmatrix} \zeta_{1,1} & \zeta_{1,2} \\ 0 & \zeta_{2,2} \end{bmatrix}.$$

Assume further that $\zeta_{1,0} = \zeta_{2,0}$. In this case, using the results developed in Appendix B.1, the dynamics of F_t can be rewritten:

$$\begin{aligned} F_{1,t} &= \mu_1\nu_1 + \mu_1\zeta_{11,F}F_{1,t-1} + \sigma_{1,t}\varepsilon_{1,t} \\ F_{2,t} &= \mu_2\nu_2 + \zeta_{12,F}F_{1,t-1} + \mu_2\zeta_{22,F}F_{2,t-1} + \tilde{\sigma}_{2,t}\tilde{\varepsilon}_{2,t}, \end{aligned}$$

where $\tilde{\varepsilon}_t = [\varepsilon_{1,t}, \tilde{\varepsilon}_{2,t}]'$ is a martingale difference sequence with identity covariance matrix and where:

$$\begin{aligned} \sigma_{1,t} &= \mu_1\sqrt{\nu_1 + 2\zeta_{11,F}F_{1,t-1}} \\ \tilde{\sigma}_{2,t} &= \mu_2\sqrt{\nu_2 + 2\zeta_{22,F}F_{2,t-1} + 2\zeta_{12,F}F_{1,t-1}}. \end{aligned}$$

Let us use the notation $\mu_i\zeta_{ii,F} = \rho_i$ for $i \in \{1,2\}$ and let us assume that (i) $1-\rho_1 = \mu_1\nu_1 = \mu_2\nu_2$ and that $\rho_1 - \rho_2 = \mu_2\zeta_{12,F}$. We get:

$$\left\{ \begin{aligned} F_{1,t} - 1 &= \rho_1(F_{1,t-1} - 1) + \sigma_{1,t}\varepsilon_{1,t} \\ F_{2,t} &= 1 - \rho_1 + \rho_1 F_{1,t-1} + \rho_2(F_{2,t-1} - F_{1,t-1}) + \tilde{\sigma}_{2,t}\tilde{\varepsilon}_{2,t}, \end{aligned} \right.$$

Defining $\varepsilon_{2,t} = \dfrac{\tilde{\sigma}_{2,t}\tilde{\varepsilon}_{2,t} - \rho_1\sigma_{1,t}\varepsilon_{1,t}}{\sqrt{\tilde{\sigma}^2_{2,t} + \sigma^2_{1,t}}}$ and $\sigma_{2,t} = \sqrt{\tilde{\sigma}^2_{2,t} + \sigma^2_{1,t}}$ leads to System (23).

Note that we have:

$$\mathbb{V}ar(F_{1,t}) = \frac{\mu_1^2(\nu_1 + 2\zeta_{11,F})}{1-\rho_1^2} = \frac{\mu_1(1-\rho_1)}{1-\rho_1^2} + 2\frac{\mu_1\rho_1}{1-\rho_1^2} = \frac{\mu_1}{1-\rho_1}. \tag{a.4}$$

C Pricing formulas

C.1 Generic pricing formulas

C.1.1 Pricing of $\exp(u'X_{t+h})$ and $v'X_{t+h}$ (settled at date $t+h$)

Proposition 5. *Under Assumptions 1 to 3, the date-t price $p(u,h,X_t)$ of the payoff $\exp(u'X_{t+h})$, that is settled at date $t+h$ is given by $\exp(\mathscr{A}_h'(u)F_t + \mathscr{B}_h'(u)N_t + \mathscr{C}_h'(u)N_{t-1} + \mathscr{D}_h(u))$, where:*

$$\begin{cases} \mathscr{A}_{h+1}(u) & = A(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - A(\delta_A,\delta_B,\delta_C) - \eta_{1,A} \\ \mathscr{B}_{h+1}(u) & = B(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - B(\delta_A,\delta_B,\delta_C) - \eta_{1,B} \\ \mathscr{C}_{h+1}(u) & = C(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - C(\delta_A,\delta_B,\delta_C) - \eta_{1,C} \\ \mathscr{D}_{h+1}(u) & = D(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - D(\delta_A,\delta_B,\delta_C) - \eta_0 + \mathscr{D}_h(u) \end{cases}$$

with $\mathscr{A}_0(u) = u_A$, $\mathscr{B}_0(u) = u_B$, $\mathscr{C}_0(u) = u_C$, $\mathscr{D}_0(u) = 0$ and where $u = (u_A', u_B', u_C')'$, u_A being of dimension n_F while u_B and u_C are of dimension J.

Proof. This proposition is clearly satisfied for $h = 0$. Assume that, for a given $h \geq 0$ and for all u and X_t, we have $p(u,h,X_t) = \exp(\mathscr{A}_h(u)'F_t + \mathscr{B}_h(u)'F_t + \mathscr{C}_h(u)'F_t + \mathscr{D}_h(u))$, then

$$\begin{aligned} & p(u,h+1,X_t) \\ = \; & E(M_{t,t+1}p(u,h,X_{t+1})|\Omega_t) \\ = \; & E\left(M_{t,t+1}\exp(\mathscr{A}_h(u)'F_{t+1} + \mathscr{B}_h(u)'N_{t+1} + \mathscr{C}_h(u)'N_t + \mathscr{D}_h(u))|\Omega_t\right) \\ = \; & E\left(\exp((\mathscr{A}_h(u)+\delta_A)'F_{t+1} + (\mathscr{B}_h(u)+\delta_B)'N_{t+1} + (\mathscr{C}_h(u)+\delta_C)'N_t)|\Omega_t\right) \times \\ & \exp(-A(\delta)'F_t - B(\delta)'N_t - C(\delta)'N_{t-1} - D(\delta) + \mathscr{D}_h(u) - \eta_0 - \eta_1'X_t) \\ = \; & \exp(\left[A(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - A(\delta_A,\delta_B,\delta_C) - \eta_{1,A}\right]' F_t) \times \\ & \exp([B(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - B(\delta_A,\delta_B,\delta_C) - \eta_{1,B}]' N_t) \times \\ & \exp(\left[C(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - C(\delta_A,\delta_B,\delta_C) - \eta_{1,C}\right]' N_{t-1}) \times \\ & \exp(D(\mathscr{A}_h(u)+\delta_A, \mathscr{B}_h(u)+\delta_B, \mathscr{C}_h(u)+\delta_C) - D(\delta) + \mathscr{D}_h(u) - \eta_0), \end{aligned}$$

which leads to the result. □

Corollary 1. *Under Assumptions 1 to 3, the date-t price of the payoff $v'X_{t+h}$, conditional on $X_t = x$, with payoff settlement at date $t+h$, is given by:*

$$\Pi(v,h,x) = v'\nabla_u p(u,h,x)\big|_{u=0}, \qquad \text{(a.5)}$$

where $p(u,h,x)$ is defined in Proposition 5 and where ∇_u denotes the Jacobian operator with *respect to the first argument of the function.*

Let us denote by $\mathbf{0}_{r\times c}$ and $\mathbf{1}_{r\times c}$ the matrices of dimensions $r\times c$ filled with 0 and 1, respectively. In addition, let e_j denote the j^{th} row vector of the identify matrix of dimension $J\times J$. Using the previous corollary with $v' = [\mathbf{0}_{1\times n_F}, e_j, \mathbf{0}_{1\times J}]$ and $v' = [\mathbf{0}_{1\times n_F}, \mathbf{0}_{1\times J}, e_j]$ respectively results in the prices of the payoffs $N_{j,t+h}$ and $N_{j,t+h-1}$ (settled at date $t+h$). With $v' = [\mathbf{0}_{1\times n_F}, \mathbf{1}_{1\times J}, \mathbf{0}_{1\times J}]$ and $v' = [\mathbf{0}_{1\times n_F}, \mathbf{0}_{1\times J}, \mathbf{1}_{1\times J}]$, we respectively get the prices of the payoffs N^*_{t+h} and N^*_{t+h-1}.

Corollary 2. *Under Assumptions 1 to 3, the date-t price of the payoff* $\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}$*, conditional on* $X_t = x$*, with payoff settlement at date* $t+h$*, is given by:*

$$\begin{aligned} g(a,b,y,h,x) &= \mathbb{E}_t\left(M_{t,t+h}\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}|X_t = x\right) \\ &= \frac{p(a,h,x)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im\left[p(a+ivb,h,x)\exp(-ivy)\right]}{v}dv, \end{aligned} \tag{a.6}$$

where $Im(z)$ *denotes the imaginary part of the complex number* z*.*

This result is proved in Duffie et al. (2000). Note that the formula for $g(a,b,y,h,x)$ is quasi explicit since it only involves a simple (one-dimensional) integration.

Corollary 3. *Under Assumptions 1 to 3, the date-t price of the payoff* $a'X_{t+h}\mathbb{1}_{\{b'X_{t+h}<y\}}$*, conditional on* $X_t = x$*, with payoff settlement at date* $t+h$*, is given by:*

$$\Gamma(a,b,y,h,x) = a'\nabla_u g(u,b,y,h,x)\big|_{u=0}. \tag{a.7}$$

C.1.2 Pricing of $\exp(u_1'X_{t+1}+\dots+u_1'X_{t+h-1}+u_2'X_{t+h})$ (settled at date $t+h$)

Proposition 6. *Using the notation* $\mathbf{u} = \{u_1,u_2\}$*, the date-t price* $\tilde{p}(\mathbf{u},h,X_t)$ *of the payoff*

$$\exp(u_1'X_{t+1}+\dots+u_1'X_{t+h-1}+u_2'X_{t+h}), \quad \textit{for } h>1$$

and of $\exp(u_2'X_{t+1})$ *for* $h=1$*, settled at date* $t+h$*, is given by* $\exp(\widetilde{\mathscr{A}_h}'(\mathbf{u})F_t + \widetilde{\mathscr{B}_h}'(\mathbf{u})N_t + \widetilde{\mathscr{C}_h}'(\mathbf{u})N_{t-1} + \widetilde{\mathscr{D}_h}(\mathbf{u}))$*, where:*

$$\begin{cases} \widetilde{\mathscr{A}_{h+1}}(\mathbf{u}) &= A(\widetilde{\mathscr{A}_h}(\mathbf{u})+\delta_A+u_{1,A}, \widetilde{\mathscr{B}_h}(\mathbf{u})+\delta_B+u_{1,B}, \widetilde{\mathscr{C}_h}(\mathbf{u})+\delta_C+u_{1,C}) - A(\delta_A,\delta_B,\delta_C) - \eta_{1,A} \\ \widetilde{\mathscr{B}_{h+1}}(\mathbf{u}) &= B(\widetilde{\mathscr{A}_h}(\mathbf{u})+\delta_A+u_{1,A}, \widetilde{\mathscr{B}_h}(\mathbf{u})+\delta_B+u_{1,B}, \widetilde{\mathscr{C}_h}(\mathbf{u})+\delta_C+u_{1,C}) - B(\delta_A,\delta_B,\delta_C) - \eta_{1,B} \\ \widetilde{\mathscr{C}_{h+1}}(\mathbf{u}) &= C(\widetilde{\mathscr{A}_h}(\mathbf{u})+\delta_A+u_{1,A}, \widetilde{\mathscr{B}_h}(\mathbf{u})+\delta_B+u_{1,B}, \widetilde{\mathscr{C}_h}(\mathbf{u})+\delta_C+u_{1,C}) - C(\delta_A,\delta_B,\delta_C) - \eta_{1,C} \\ \widetilde{\mathscr{D}_{h+1}}(\mathbf{u}) &= D(\widetilde{\mathscr{A}_h}(\mathbf{u})+\delta_A+u_{1,A}, \widetilde{\mathscr{B}_h}(\mathbf{u})+\delta_B+u_{1,B}, \widetilde{\mathscr{C}_h}(\mathbf{u})+\delta_C+u_{1,C}) - D(\delta_A,\delta_B,\delta_C) - \eta_0 + \widetilde{\mathscr{D}_h}(\mathbf{u}), \end{cases}$$

with $\widetilde{\mathscr{A}_1}(\mathbf{u}) = \mathscr{A}_1(u_2)$, $\widetilde{\mathscr{B}_1}(\mathbf{u}) = \mathscr{B}_1(u_2)$, $\widetilde{\mathscr{C}_1}(\mathbf{u}) = \mathscr{C}_1(u_2)$, $\widetilde{\mathscr{D}_1}(\mathbf{u}) = \mathscr{D}_1(u_2)$.

Proof. This proposition is clearly satisfied for $h = 1$. Assume that, for a given $h \geq 1$ and for all $\mathbf{u}$ and X_t, we have $\tilde{p}(\mathbf{u}, h, X_t) = \exp(\widetilde{\mathscr{A}}_h(\mathbf{u})'F_t + \widetilde{\mathscr{B}}_h(\mathbf{u})'F_t + \widetilde{\mathscr{C}}_h(\mathbf{u})'F_t + \widetilde{\mathscr{D}}_h(\mathbf{u}))$, then

$$\begin{aligned}
& \tilde{p}(\mathbf{u}, h+1, X_t) \\
= \; & E(M_{t,t+1} \exp(u_1' X_{t+1}) \tilde{p}(\mathbf{u}, h, X_{t+1}) | \Omega_t) \\
= \; & E\left(\exp((\widetilde{\mathscr{A}}_h(\mathbf{u}) + \delta_A + u_{1,A})' F_{t+1} + (\widetilde{\mathscr{B}}_h(\mathbf{u}) + \delta_B + u_{1,B})' N_{t+1} + (\widetilde{\mathscr{C}}_h(\mathbf{u}) + \delta_C + u_{1,C})' N_t) | \Omega_t\right) \times \\
& \exp(-A(\delta)' F_t - B(\delta)' N_t - C(\delta)' N_{t-1} - D(\delta) + \widetilde{\mathscr{D}}_h(\mathbf{u}) - \eta_0 - \eta_1' X_t) \\
= \; & \exp(\left[A(\widetilde{\mathscr{A}}_h(\mathbf{u}) + \delta_A + u_{1,A}, \widetilde{\mathscr{B}}_h(\mathbf{u}) + \delta_B + u_{1,B}, \widetilde{\mathscr{C}}_h(\mathbf{u}) + \delta_C + u_{1,C}) - A(\delta_A, \delta_B, \delta_C) - \eta_{1,A}\right]' F_t) \times \\
& \exp(\left[B(\widetilde{\mathscr{A}}_h(\mathbf{u}) + \delta_A + u_{1,A}, \widetilde{\mathscr{B}}_h(\mathbf{u}) + \delta_B + u_{1,B}, \widetilde{\mathscr{C}}_h(\mathbf{u}) + \delta_C + u_{1,C}) - B(\delta_A, \delta_B, \delta_C) - \eta_{1,B}\right]' N_t) \times \\
& \exp(\left[C(\widetilde{\mathscr{A}}_h(\mathbf{u}) + \delta_A + u_{1,A}, \widetilde{\mathscr{B}}_h(\mathbf{u}) + \delta_B + u_{1,B}, \widetilde{\mathscr{C}}_h(\mathbf{u}) + \delta_C + u_{1,C}) - C(\delta_A, \delta_B, \delta_C) - \eta_{1,C}\right]' N_{t-1}) \times \\
& \exp(D(\widetilde{\mathscr{A}}_h(\mathbf{u}) + \delta_A + u_{1,A}, \widetilde{\mathscr{B}}_h(\mathbf{u}) + \delta_B + u_{1,B}, \widetilde{\mathscr{C}}_h(\mathbf{u}) + \delta_C + u_{1,C}) - D(\delta) + \widetilde{\mathscr{D}}_h(\mathbf{u}) - \eta_0),
\end{aligned}$$

which leads to the result. □

C.2 CDS formula

Let's rewrite eq. (7):

$$S^{CDS}_{j,t,h} \;=\; q(1-RR)\frac{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(d_{j,i,t+k} - d_{j,i,t+k-1})\right\}}{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(1 - d_{j,i,t+k})\right\}}.$$

Using eqs. (8) and (9), we get:

$$S^{CDS}_{j,t,h} \approx q(1-RR)\frac{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}[N_{j,t+k} - N_{j,t+k-1}]\right\}}{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(I_j - N_{j,t+k})\right\}}.$$

C.3 Tranche products formula

Let's rewrite eq. (11):

$$\begin{aligned}
& \mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}\frac{\tilde{N}_{t+k} - \tilde{N}_{t+k-1}}{\bar{b} - \bar{a}}\mathbb{1}_{\{\bar{a} < \tilde{N}_{t+k} \leq \bar{b}\}}\right\} \\
= \; & U^{TDS}_{t,h}(a,b) + \mathbb{E}_t\left\{\frac{S^{TDS}_{t,h}(a,b)}{q}\sum_{k=1}^{qh} M_{t,t+k}\left(\mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{a}\}} + \frac{\bar{b} - \tilde{N}_{t+k}}{\bar{b} - \bar{a}}\mathbb{1}_{\{\bar{a} < \tilde{N}_{t+k} \leq \bar{b}\}}\right)\right\},
\end{aligned}$$

where $\overline{a} = a\frac{\tilde{I}}{1-RR}$ and $\overline{b} = b\frac{\tilde{I}}{1-RR}$. We obtain:

$$S_{t,h}^{TDS}(a,b) = \frac{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}(\tilde{N}_{t+k}-\tilde{N}_{t+k-1})\left(\mathbb{1}_{\{\tilde{N}_{t+k}\leq\overline{b}\}} - \mathbb{1}_{\{\tilde{N}_{t+k}\leq\overline{a}\}}\right)\right\} - (\overline{b}-\overline{a})U_{t,h}^{TDS}(a,b)}{\mathbb{E}_t\left\{\sum_{k=1}^{qh} M_{t,t+k}\left((\overline{b}-\overline{a})\mathbb{1}_{\{\tilde{N}_{t+k}\leq\overline{a}\}} + (\overline{b}-\tilde{N}_{t+k})\left(\mathbb{1}_{\{\tilde{N}_{t+k}\leq\overline{b}\}} - \mathbb{1}_{\{\tilde{N}_{t+k}\leq\overline{a}\}}\right)\right)\right\}}.$$

C.4 Proof of Prop. 2

Let us introduce the log price-dividend ratio defined by $z_t = \log(P_t/D_t)$ and let us denote by $\bar{z}$ its marginal expectation. The following proposition is based on the log-linearization proposed by Campbell (1993).

Proposition 7. *if $z_t - \bar{z}$ is relatively small, then the stock return r_{t+1}^s can be approximated by*

$$r_{t+1}^s = \log\left(\frac{P_{t+1}+D_{t+1}}{D_t}\right) \approx \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{d,t+1}, \tag{a.8}$$

where $g_{d,t}$ is the log growth rate of dividends and where

$$\left\{\begin{array}{rcl} \kappa_1 & = & \dfrac{\exp(\bar{z})}{1+\exp(\bar{z})} \\ \kappa_0 & = & \log(1+\exp(\bar{z})) - \kappa_1\bar{z}. \end{array}\right. \tag{a.9}$$

Proof. We have:

$$\begin{aligned} r_{t+1}^s &= \log\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right) \\ &= \log(P_{t+1}+D_{t+1}) - \log(P_t) + \log(P_{t+1}) - \log(P_{t+1}) + \\ &\quad \log(D_{t+1}) - \log(D_{t+1}) + \log(D_t) - \log(D_t) \\ &= z_{t+1} - z_t + g_{d,t+1} + \log\left(1 + \frac{D_{t+1}}{P_{t+1}}\right). \end{aligned} \tag{a.10}$$

Besides:

$$\begin{aligned} \log[1+D_{t+1}/P_{t+1}] &= \log[1+\exp(-z_{t+1})] \\ &\approx \log[1+\exp(-\bar{z})\{1-(z_{t+1}-\bar{z})\}] \\ &\approx \log[1+\exp(-\bar{z}) - \exp(-\bar{z})(z_{t+1}-\bar{z})] \\ &\approx \log[1+\exp(-\bar{z})] - \frac{z_{t+1}-\bar{z}}{1+\exp(\bar{z})}. \end{aligned}$$

Therefore:

$$\begin{aligned} r^s_{t+1} &\approx z_{t+1} - z_t + g_{d,t+1} + \log[1+\exp(-\bar{z})] - \frac{z_{t+1}-\bar{z}}{1+\exp(\bar{z})} \\ &\approx \log[1+\exp(-\bar{z})] + \frac{\bar{z}}{1+\exp(\bar{z})} + \frac{\exp(\bar{z})}{1+\exp(\bar{z})} z_{t+1} - z_t + g_{d,t+1} \\ &\approx \log[1+\exp(\bar{z})] - \frac{\exp(\bar{z})}{1+\exp(\bar{z})}\bar{z} + \frac{\exp(\bar{z})}{1+\exp(\bar{z})} z_{t+1} - z_t + g_{d,t+1}, \end{aligned}$$

which leads to eq. (a.8). □

Assume that z_t is affine in X_t, i.e.:

$$z_t = A_0 + A_1' X_t. \tag{a.11}$$

Substituting for z_t in eq. (a.8) leads to eq. (13). Let us now determine the constraints that should be satisfied by A_0 and A_1. As for any asset, the returns of stocks have to satisfy the Euler equation:

$$0 = \log \mathbb{E}_t(M_{t,t+1} \exp(r^s_{t+1})). \tag{a.12}$$

Using eqs. (5) and (13), we obtain:

$$\begin{aligned} &M_{t,t+1}\exp(r^s_{t+1})) = \qquad\qquad (a.13)\\ &\exp(\kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} - \eta_0 - \psi_0(\delta) + (\kappa_1 A_1 + \mu_{d,1} + \delta)' X_{t+1} - (A_1 + \eta_1 + \psi_1(\delta))' X_t). \end{aligned}$$

Eqs. (a.13) and (a.12) are satisfied if:

$$\left\{ \begin{array}{rcl} \kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} - \eta_0 - \psi_0(\delta) + \psi_0(\kappa_1 A_1 + \mu_{d,1} + \delta) &=& 0 \\ \psi_1(\kappa_1 A_1 + \mu_{d,1} + \delta) - (A_1 + \eta_1 + \psi_1(\delta)) &=& 0, \end{array} \right. \tag{a.14}$$

which proves Prop. 2.

C.5 Stock option pricing

If z_t and $g_{d,t}$ are affine in X_t (as in eqs. (12) and (a.11)), then eq. (16) implies that P_{t+h}/P_t is exponential affine in X_t.

Let us introduce function φ defined by:

$$(u,h,X_t) \to \varphi(u,h,X_t) = \mathbb{E}_t\left(M_{t,t+h}\exp\left(u\log\left(\frac{P_{t+h}}{P_t}\right)\right)\right).$$

Using eq. (16) and Prop. 6, one can find functions $a_h^s(\bullet)$ and $b_h^s(\bullet)$ that are such that:

$$\varphi(u,h,X_t) = \exp(a_h^s(u) + b_h^s(u)'X_t).$$

Replacing $p(a,h,x)$ by $\varphi(a,h,x)$ in Corollary 2 provides formula to compute

$$\mathbb{E}_t\left(M_{t,t+h}\exp\left[a\log\left(\frac{P_{t+h}}{P_t}\right)\right]\mathbb{1}_{\left\{b\log\left(\frac{P_{t+h}}{P_t}\right)<y\right\}}\middle| X_t = x\right).$$

Let us denote by $g^*(a,b,y,h,x)$ the previous expression. With this notation, the price of a put option (eq. 17) reads:

$$\begin{aligned}
& \mathbb{E}_t\left(M_{t,t+h}(K-P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right) \\
= \; & K\mathbb{E}_t\left(M_{t,t+h}\mathbb{1}_{\{r^*_{t+1}+\dots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\
& -P_t\mathbb{E}_t\left(M_{t,t+h}\exp(r^*_{t+1}+\dots+r^*_{t+h})\mathbb{1}_{\{r^*_{t+1}+\dots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\
= \; & Kg^*(0,1,\log(K)-\log P_t,h,X_t) - P_t g^*(1,1,\log(K)-\log P_t,h,X_t).
\end{aligned}$$

D Endowment economy

D.1 Proof of Prop. 3

Let us consider the following specification for Δu_t:

$$\Delta u_t = \mu_{u,0} + \mu'_{u,1}X_t + \mu'_{u,2}X_{t-1}.$$

Then, for a given $(X'_t, X'_{t-1})'$, we have:

$$\begin{aligned}
\text{eq. (20)} \quad \Leftrightarrow \quad & \mu_{u,0} + \mu'_{u,1}X_t + \mu'_{u,2}X_{t-1} \\
& = \mu_{c,0} + \mu'_{c,1}X_t + \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\left\{[\psi_1((1-\gamma)\mu_{u,1}) + (1-\gamma)\mu_{u,2}]'(X_t - X_{t-1})\right\}.
\end{aligned}$$

Therefore eq. (20) is satisfied for any $(X'_t, X'_{t-1})'$ iff

$$\begin{cases}
\frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) + \frac{1}{1-\delta}\mu_{u,2} = 0 \\
\mu_{u,1} - \mu_{c,1} - \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) - \frac{\delta}{1-\delta}\mu_{u,2} = 0 \\
\mu_{u,0} = \mu_{c,0},
\end{cases}$$

or

$$\begin{cases} \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1})+\frac{1}{1-\delta}\mu_{u,2}=0 \\ \mu_{u,1}+\mu_{u,2}-\mu_{c,1}=0 \\ \mu_{u,0}=\mu_{c,0}, \end{cases} \tag{a.15}$$

which leads to the result.

D.2 Proof of Prop. 4

Epstein and Zin (1989) have shown that when agent's preferences are as in eq. (19), the s.d.f. is given by:

$$M_{t,t+1}=\delta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{U_{t+1}^{1-\gamma}}{\mathbb{E}_t(U_{t+1}^{1-\gamma})}.$$

Therefore, we have:

$$\begin{aligned} \log M_{t,t+1} &= \log\delta-\Delta c_{t+1}+(1-\gamma)u_{t+1}-\log\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}]) \\ &= \log(\delta)-\mu_{c,0}-\mu_{c,1}'X_{t+1}+(1-\gamma)(\mu_{u,0}+\mu_{u,1}'X_{t+1}+\mu_{u,2}'X_t) \\ &\quad -\log\mathbb{E}_t(\exp[(1-\gamma)(\mu_{u,0}+\mu_{u,1}'X_{t+1}+\mu_{u,2}'X_t)]) \\ &= \log(\delta)-\mu_{c,0}-\psi_0((1-\gamma)\mu_{u,1})+[(1-\gamma)\mu_{u,1}-\mu_{c,1}]'X_{t+1}-\psi_1((1-\gamma)\mu_{u,1})'X_t, \end{aligned}$$

which leads to the result.

E Data

E.1 Credit index and tranche prices (iTraxx)

E.1.1 The iTraxx credit index and constituents

To estimate the model, we employ financial data based on the iTraxx Europe main index, an index involving 125 large European firms. iTraxx indices roll every six month. That is, every six months, a new series of the index is created with updated constituents. The previous series continues trading, although liquidity is concentrated on the on-the-run series [see Markit (2014)].

The roll consists of a series of steps which are administered by Markit. For the Markit iTraxx Europe indices, liquidity lists are formed from the trading volumes from the DTCC Trade Information Warehouse.[34] Markit then applies index rules to determine the index constituents among

[34] http://www.dtcc.com/derivatives-services/trade-information-warehouse.

the most liquid names [see e.g. Markit (2016)]. For iTraxx Europe main (the index used in this study), the final Index comprises 30 Autos & Industrials, 30 Consumers, 20 Energy, 20 TMT, 25 Financials.

Constituents of the iTraxx Europe main index must have an investment grade rating. That is, to be included in the list of constituents, entities have to be rated BBB-/Baa3/BBB- (Fitch/-Moody's/S&P) or higher. In March 2016, the median rating for the iTraxx index (series 25) is BBB+ at S&P [Société Générale (2016)].

E.1.2 Data sources and preliminary transformations

We extract spreads of iTraxx indices from Thomson Datastream. These spreads correspond to maturities of 3, 5, 7 and 10 years. iTraxx tranche prices come from the Markit website.[35] For each maturity, we use prices associated with the following tranches: 0%-3%, 3%-6%, 6%-9%, 9%-12% and 12%-22%. We do not use prices associated with the super-senior tranche (22%-100%) as well as prices associated with the 10-year maturity given the very low liquidity of these contracts. Note also that, for liquidity reasons, our Markit data do not cover all dates in our sample. In particular, we do not have tranche prices before January 2007 and after March 2013.

Because each index roll features fixed maturity dates, market prices are not of the "constant-maturity" type. To deal with this, for each considered maturity and for (i) each date and (ii) each pair of attachment/detachment points, we look for the tranche price whose residual maturity is the closest to the considered one. If the residual maturity of the resulting tranche is not in a ± 1 year window around the targeted maturity, no price is reported.

E.2 Equity options (EURO STOXX 50)

Equity put options are far out-of-the-money options written on the EURO STOXX 50 index. We consider two maturities, 6 and 12 months, and strikes equal to 70% of the current value of the index. That is, the payoffs of these options become strictly positive in case of a fall of the index by more than 30%. Note such option prices are not directly available on Thomson Datastream; option prices reported on those database are for contracts with standardized maturity dates and strikes. We compute the prices of our out-of-money options by applying interpolation splines on available data, in both the time and strikes dimensions. Following market convention, we convert our put option prices into implied volatilities using the Black-Scholes formula.

[35] `http://www.creditfixings.com/CreditEventAuctions/itraxx.jsp`. For each date, maturity and tranche, we convert all quotes into an equivalent running spread with no upfront payment by using the risky duration approach [see e.g. O'Kane and Sen (2003), D'Amato and Gyntelberg (2005) or Morgan Stanley (2011)].

– Online Appendix –

Disastrous Defaults

Christian GOURIÉROUX, Alain MONFORT, Sarah MOUABBI and Jean-Paul RENNE

A.1 Conditional and unconditional moments of $[F_t', n_t']'$

In this appendix, we derive conditional and unconditional moments of $[F_t', n_t']'$ when this vector follows the dynamics described in Subsection B.1 together with Assumption 1.

We have:

$$\mathbb{E}_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right) = \left[\begin{array}{c} \mu_F \\ \gamma+\beta'\mu_F \end{array}\right] + \left[\begin{array}{cc} \Phi_{FF} & \Phi_{Fn} \\ \beta'\Phi_{FF} & \beta'\Phi_{Fn}+c' \end{array}\right]\left[\begin{array}{c} F_t \\ n_t \end{array}\right]$$

and

$$\begin{aligned}
\mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right) &= \mathbb{E}_t\left(\mathbb{V}ar\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\middle|\Omega_t^*\right)\right) + \mathbb{V}ar_t\left(\mathbb{E}\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\middle|\Omega_t^*\right)\right) \\
&= \mathbb{E}_t\left(\left[\begin{array}{cc} 0 & 0 \\ 0 & \mathrm{diag}(\beta'F_{t+1}+c'n_t+\gamma) \end{array}\right]\right) + \mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ \beta'F_{t+1}+c'n_t+\gamma \end{array}\right]\right) \\
&= \left[\begin{array}{cc} 0_{n_F\times n_F} & 0_{n_F\times J} \\ 0_{J\times n_F} & \mathrm{diag}(\beta'(\mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t)+c'n_t+\gamma) \end{array}\right] + \\
&\quad \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\mathrm{diag}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right)\left[\begin{array}{cc} I_{n_F} & \beta \end{array}\right] \\
&= \left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\mathrm{diag}(\beta'(\mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t)+c'n_t+\gamma)\left[\begin{array}{cc} 0_{J\times n_F} & I_J \end{array}\right] \\
&\quad \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\mathrm{diag}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right)\left[\begin{array}{cc} I_{n_F} & \beta \end{array}\right].
\end{aligned}$$

Using the fact that, for any n-dimensional vector a:

$$vec(diag(a)) = \sum_{i=1}^{n} vec(diag[e_ie_i'a]) = \sum_{i=1}^{n} vec(e_ie_i')a_i = \underbrace{\left(\sum_{i=1}^{n} vec(e_ie_i')e_i'\right)}_{=:S_n} a,$$

we obtain:

$$
\begin{aligned}
& vec\left(\mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right)\right) \\
= & \left(\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\otimes\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\right) S_J(\beta'(\mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t)+c'n_t+\gamma)+ \\
& \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\otimes\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right) \\
= & \left(\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\otimes\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\right) S_J(\beta'\mu_F+\gamma)+\left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\otimes\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F}\mu_F^{var}+ \\
& \left\{\left(\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\otimes\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\right) S_J\beta'\Phi_{FF}+\left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\otimes\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F}\Phi_{FF}^{var}\right\} F_t+ \\
& \left\{\left(\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\otimes\left[\begin{array}{c} 0_{n_F\times J} \\ I_J \end{array}\right]\right) S_J\beta'\Phi_{Fn}+\left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\otimes\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F}\Phi_{Fn}^{var}\right\} n_t.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
& vec\left(\mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right)\right) \\
= & \left[\begin{array}{c}
S_{n_F}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right) \\
(I_{n_F}\otimes\beta')S_{n_F}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right) \\
(\beta'\otimes I_{n_F})S_{n_F}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right) \\
S_J(\beta'(\mu_F+\Phi_{FF}F_t+\Phi_{Fn}n_t)+c'n_t+\gamma)+(\beta'\otimes\beta')S_{n_F}\left(\mu_F^{var}+\Phi_{FF}^{var}F_t+\Phi_{Fn}^{var}n_t\right)
\end{array}\right] \\
= & \left[\begin{array}{c}
S_{n_F}\mu_F^{var} \\
(I_{n_F}\otimes\beta')S_{n_F}\mu_F^{var} \\
(\beta'\otimes I_{n_F})S_{n_F}\mu_F^{var} \\
S_J(\beta'\mu_F+\gamma)+(\beta'\otimes\beta')S_{n_F}\mu_F^{var}
\end{array}\right]+ \\
& \left[\begin{array}{c}
S_{n_F}\Phi_{FF}^{var} \\
(I_{n_F}\otimes\beta')S_{n_F}\Phi_{FF}^{var} \\
(\beta'\otimes I_{n_F})S_{n_F}\Phi_{FF}^{var} \\
S_J\beta'\Phi_{FF}+(\beta'\otimes\beta')S_{n_F}\Phi_{FF}^{var}
\end{array}\right] F_t+\left[\begin{array}{c}
S_{n_F}\Phi_{Fn}^{var} \\
(I_{n_F}\otimes\beta')S_{n_F}\Phi_{Fn}^{var} \\
(\beta'\otimes I_{n_F})S_{n_F}\Phi_{Fn}^{var} \\
S_J\beta'\Phi_{Fn}+(\beta'\otimes\beta')S_{n_F}\Phi_{Fn}^{var}
\end{array}\right] n_t.
\end{aligned}
$$

Hence, both $vec\left(\mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right)\right)$ and $\mathbb{E}_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right)$ are affine functions of $\left[\begin{array}{c} F_t \\ n_t \end{array}\right]$. Let us introduce the obvious notations:

$$\begin{aligned} \mathbb{E}_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) &= \mu_E + \Phi_E \begin{bmatrix} F_t \\ n_t \end{bmatrix} \\ vec\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)\right) &= \mu_V + \Phi_V \begin{bmatrix} F_t \\ n_t \end{bmatrix}. \end{aligned}$$

Assuming that $\begin{bmatrix} F_t \\ n_t \end{bmatrix}$ is covariance-stationary, we have:

$$\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \mathbb{E}\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) = \mu_E + \Phi_E \mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right),$$

which leads to

$$\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = (I - \Phi_E)^{-1}\mu_E.$$

We also have:

$$\mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \mathbb{E}\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) + \mathbb{V}ar\left(\mathbb{E}_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right).$$

Using that

$$\begin{cases} \mathbb{E}\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = \mathbb{E}\left(\mu_V + \Phi_V \begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) \\ \mathbb{V}ar\left(\mathbb{E}_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = \mathbb{V}ar\left(\mu_E + \Phi_E \begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \Phi_E \mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\Phi_E', \end{cases}$$

we get:

$$vec\left(\mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = (I - \Phi_E \otimes \Phi_E)^{-1}\left[\mu_V + \Phi_V \mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right].$$

A.2 Maximum Sharpe ratio between dates t and $t+h$

The maximum Sharpe ratio of an investment realized between dates t to $t+h$ is given by [see Hansen and Jagannathan (1991)]:

$$\mathscr{M}_{t,t+h} = \frac{\sqrt{\mathbb{V}ar_t(M_{t,t+h})}}{\mathbb{E}_t(M_{t,t+h})}.$$

We have:

$$\begin{aligned} M_{t,t+h} &= \exp\left(h\mu_{0,m} + \mu_{2,m}' X_t\right) \times \\ &\quad \exp\left([\mu_{1,m} + \mu_{2,m}]' X_{t+1} + \cdots + [\mu_{1,m} + \mu_{2,m}]' X_{t+h-1} + \mu_{1,m}' X_{t+h}\right) \end{aligned}$$

Therefore:

$$\mathscr{M}_{t,t+h} = \frac{\sqrt{\Theta_{t,h}(2[\mu_{1,m} + \mu_{2,m}], 2\mu_{1,m}) - \Theta_{t,h}(\mu_{1,m} + \mu_{2,m}, \mu_{1,m})^2}}{\Theta_{t,h}(\mu_{1,m} + \mu_{2,m}, \mu_{1,m})},$$

where

$$\Theta_{t,h}(u,v) = \mathbb{E}_t(\exp(u' X_{t+1} + \cdots + u' X_{t+h-1} + v' X_{t+h})).$$

When X_t is an affine process, $\Theta_{t,h}(u,v)$ can be computed in closed-form by using recursive formulas as in Prop. 6 (with $\delta \equiv \eta \equiv 0$)