

Geographical diversification in annuity portfolios*

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Abstract

This paper studies the problem of an insurance company that has to decide whether to expand her portfolio of policies selling contracts written on a foreign population. We propose a parsimonious continuous-time model for longevity risk, that captures the dependence across different ages in two populations and evaluate the diversification gains due to the international expansion. We present a calibrated example, based on annuity portfolios of UK and Italian males aged 65-75. We show that diversification gains, evaluated as the reduction in the portfolio risk margin following the international expansion, can be non-negligible, in particular when interest rates are low. We describe how the expansion can obtain through a swap, instead of opening a foreign affiliate.

Keywords: geographical diversification; life insurance; risk management; multi-population mortality; longevity risk modeling.

JEL classification: F23, F65, G15, G22.

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1 Introduction

In the last twenty years, insurance companies have been expanding internationally, via operating subsidiaries in different countries or via cross-border mergers and acquisitions.¹ Reinsurance companies have always been more geographically diversified than insurance companies (Cummins and Xie (2008)), because their portfolios are more easily disconnected from the geographical localization of their branches. As a result, the largest insurers and reinsurers in most cases concentrate the bulk of their activities in foreign countries. Evidence is provided for instance by Outreville (2008).

The connection between internationalization and profitability has been the subject of some recent studies. Outreville (2012) finds that, in the reinsurance industry, the level of international diversification affects performance, but the relationship is non-linear. In life insurance, Biener et al. (2015) shows that costs of coordination and organization of complex international structures may offset the potential benefits from internationalization, resulting in a negative relationship between performance and internationalization. Cummins et al. (1999) argues that geographical diversification is a primary determinant of mergers and acquisitions in the US insurance industry, because geographically diversified firms were more likely to be the target of acquisitions.

The connection between internationalization and risk or capital requirements reduction instead has been very little explored. This paper addresses a typical diversification gain from internationalization of life insurance or reinsurance companies, namely longevity risk pooling across populations. We consider the situation of an insurer who is facing the choice of expanding her portfolio of liabilities (for simplicity, an annuity portfolio held by policyholders of different ages) either in her domestic market or in a foreign one. We want to determine both the normalized amount of longevity risk reduction, which we call diversification index, and its impact in terms of risk margin for regulatory purposes.

Why is international diversification of life insurance portfolios beneficial in terms of risk? Because, even if – in expectation – longevity has been steadily increasing on a worldwide scale, longevity risk of different populations may differ. That risk is due to unexpected fluctuations in mortality rates, and the latter may be non-perfectly correlated across countries. As a result, pooling portfolios of policies written on the lives of different populations allows to diversify longevity risks. The previous literature in longevity risk however has either taken the correlation of lives of different generations in different populations as exogenous, without calibrating it, or has used it in order to assess the basis risk implied in standardized longevity transfer transactions (see for instance Li and Hardy (2011) or Coughlan et al. (2011)). Up to our knowledge, the effects of non-perfect correlation in the co-movements of mortality rates of different populations, and generations within them, on the risk and capital requirements of internationally diversified portfolios of life insurers have been neglected. That is because assessing whether diversifications gains are sizable is by no means a trivial task.

To fill up the hole in the literature, we first set up a model for the joint mortality dynamics of policyholders in different countries. We consider a domestic and a foreign market, and heterogeneous cohorts in each market. We setup a parsimonious model of stochastic mortality intensity, that extends the deterministic Gompertz intensity,

¹In Europe, for instance, the creation of a common regulation framework in the middle of the Nineties gave rise to a wave of international expansions and M& A operations.

a benchmark in the classical modelling of mortality arrival rates. We show that the model fits well the observed mortality rates and is able to capture the imperfect correlations observed across ages and populations. We indeed apply the model to estimate the joint dynamics of the mortality rates of UK and Italian males aged between 65 and 75. We estimate a whole correlation matrix across the eleven cohorts of the two different populations.

Based on the model, which captures the level of correlation through a single parameter, we define a similarity/diversification index between two annuity portfolios written on two different populations. This index is a normalized measure of the degree of diversification of insurance portfolios. We then compute a non-normalized measure of the risk effects of international expansion, the risk-margin reduction. Reminiscent of the Solvency II regulation framework, we compute the risk margin of a life insurance portfolio as its value-at-risk (VaR). The difference between the risk margin of a life insurer who expands its portfolio domestically and, all others equal, diversifies it internationally, provides us with a (non-normalized, or dollar-based) measure of the diversification benefits. Our application, that considers the choice of a UK-based annuity provider, shows that the risk margin can reduce up to 2% as a proportion of the actuarial value, in the case of a foreign expansion to Italy. Also, longevity risk mitigation effects are shown to be more sizable when the interest rate is lower. Last, we use the diversification index and the risk margin reduction to find out the optimal portfolio mix of the life insurer. We define the latter as the liability mix that gives the smallest risk margin, and show that it corresponds also to the highest diversification index.

The diversification benefits of an international expansion, however, may happen to be counterbalanced by the costs connected to the foreign portfolio acquisition process. These costs, that are - say - the fixed costs of opening a foreign affiliate, or the fees required by the agents involved in the M & A operation, etc., may be substantial. As an alternative to a physical expansion, the insurer may obtain the same diversification benefit operating on the longevity derivatives market. Longevity derivatives, and longevity swaps in particular, are bespoke transactions between (re)insurers and funds or companies, that agree to exchange fixed cash flows and cash flows linked to the survivorship of a particular population (see Blake et al. (2006) for instance). The buyer of the protection provided by a longevity swap transfers the longevity risk linked to a given reference population to the seller, who in turn becomes exposed to such risk. In our case, the insurer can expand internationally by receiving a fixed periodical fee and paying the survivorship of the foreign cohorts. Thus, the risk margin reduction benefits of a foreign expansion can be replicated by selling protection through a swap. Even in this case, however, the costs of structuring the agreement and coping with informational asymmetries (Biffis et al. (2016)), can substantially reduce the diversification gains. Assessing these costs and comparing them with those sustained when expanding physically completes our work.

The paper unfolds as follows. Section 2 describes the problem of the insurer and the value of its portfolio in the alternative cases in which he expands domestically vs. internationally. Section 3 presents the mortality model and our diversification index measure. Section 4 provides a calibrated application, describing our model calibration procedure, which is based on RMSE minimization and Gaussian mapping, and the value of the diversification and risk margin corresponding to various portfolio choices.

Section 5 discusses the fair price of the swap that provides the best international diversification, as defined above.

2 The Insurer's Problem

We consider an Annuity Provider, or Life-Insurer, based in a generic Country (that we call *Domestic*), having a portfolio of deferred annuities written on different cohorts belonging to the Domestic Population. Let $\mathcal{X} = \{x_1, \dots, x_m\}$ be the set of annuitants' ages at time zero, and let n_i , for $i = 1, \dots, m$, be the number of annuities sold to people aged x_i . We consider a portfolio in which the oldest policyholders are 75:

$$x_i \leq 75 \quad \forall i = 1, \dots, m. \quad (1)$$

When an annuity is sold at time zero, the annuitant pays an initial premium $\Pi_i^0(0)$. After signing the contract, he will receive a series of fixed annual payments R , starting from the year-end of his 65-th birthday if $x_i < 65$, or immediately if $x_i \geq 65$, until his death, that may happen at most when he reaches a final age ω (for instance $\omega = 115$). In Europe, the life-insurance business is regulated by the Solvency II regulation, that requires insurers to value their liabilities at market value and set aside VaR-based risk margins with respect to both financial and longevity risk. To reproduce such situation, we assume that the value $\Pi^0(t)$ of the portfolio at time t is the sum of two components: the Actuarial Value of the contracts of portfolio $AV_{\Pi^0}(t)$, which is the sum of the actuarial values of each individual contract $N_i(t)$, and the Risk Margin $RM_{\Pi^0}(t)$ of the portfolio itself, i.e.

$$\Pi^0(t) = AV_{\Pi^0}(t) + RM_{\Pi^0}(t) = \sum_{i=1}^m n_i N_i(t) + RM_{\Pi^0}(t). \quad (2)$$

The portfolio risk margin $RM_{\Pi^0}(t)$ is defined as the discounted Value-at-Risk, at a certain confidence interval $\alpha \in (0, 1)$, of the unexpected portfolio's future actuarial value at a given time horizon T :

$$\begin{aligned} RM_{\Pi^0}(t) &= D(t, t+T) \cdot VaR_{\alpha}(AV_{\Pi^0}(t+T) - \mathbb{E}_t[AV_{\Pi^0}(t+T)]), \\ &= D(t, t+T) \cdot \inf\{l \in \mathbb{R}^+ : P(AV_{\Pi^0}(t+T) - \mathbb{E}_t[AV_{\Pi^0}(t+T)] > l) < 1 - \alpha\}. \end{aligned} \quad (3)$$

$$(4)$$

Let us consider an annuity contract sold at time $t \geq 0$ to an individual aged x_i . The number of years before the individual reaches age 65 is $\tau = 65 - x_i$. If $\tau \geq 0$, then the contract is a deferred annuity and its actuarial value is

$$N_i(t) = D(t, t+\tau) S_i(t, t+\tau) \left[R \sum_{u=1}^{\omega-t-\tau} D(t+\tau, t+\tau+u) S_i(t+\tau, t+\tau+u) \right]. \quad (5)$$

If $\tau < 0$, then the individual is already aged at least 65, the contract is an immediate annuity and its actuarial value is:

$$N_i(t) = R \sum_{u=1}^{\omega-t} D(t, t+u) S_i(t, t+u). \quad (6)$$

Formula (5) can be used in both cases by using the convention $\tau = \max(65 - x_i, 0)$.

2.1 Portfolio Expansion

We consider the case in which the Insurer wants to increase the size of her annuity portfolio and needs to choose between two possible strategies. The first one is just sell new contracts to her Domestic population. In this case, we denote with n'_i the number of new contracts sold to people aged x_i , with Π^D the portfolio composed of just these new annuities, and with Π^1 the total portfolio of old and new contracts. It is easy to see that the actuarial value of the new portfolio is

$$AV_{\Pi^D}(t) = \sum_{i=1}^m n'_i N_i(t), \quad (7)$$

and

$$AV_{\Pi^1}(t) = AV_{\Pi^0}(t) + AV_{\Pi^D}(t). \quad (8)$$

The value of the total portfolio Π is the sum of the actuarial value of the old portfolio, the actuarial value of the new portfolio and the risk margin of the total portfolio:

$$\Pi^1(t) = AV_{\Pi^1}(t) + RM_{\Pi^1}(t) = AV_{\Pi^0}(t) + AV_{\Pi^D}(t) + RM_{\Pi^1}(t). \quad (9)$$

The second possible strategy is to acquire a new portfolio of annuities Π^F , written on a foreign population. To compare the two strategies, we simply assume that for each age x_i the number of annuities written on people aged x_i in the foreign population is still n'_i . The actuarial value of this portfolio is

$$AV_{\Pi^F}(t) = \sum_{i=1}^m n'_i N_i^F(t), \quad (10)$$

and

$$AV_{\Pi^2}(t) = AV_{\Pi^0}(t) + AV_{\Pi^F}(t). \quad (11)$$

Moreover,

$$\Pi^2(t) = AV_{\Pi^2}(t) + RM_{\Pi^2}(t) = AV_{\Pi^0}(t) + AV_{\Pi^F}(t) + RM_{\Pi^2}(t). \quad (12)$$

Our objective is to compare these two strategies in order to study and quantify the effects of diversification in annuity portfolios.

3 Longevity Risk Modeling

We now turn to the description of the source of uncertainty that affects the value of the Insurer's portfolio: the risk of longevity, i.e. the risk that her policyholders live longer than expected. We set ourselves in the well-established continuous-time stochastic mortality setting initiated by Milevsky and Promislow (2001) that models the death of individuals as a Cox process. The time to death of an individual τ_{x_i} belonging to cohort x_i is the first jump time of a Poisson process with stochastic intensity. Let us consider two populations, each containing m different cohorts. The first population is called the *Domestic population* and the second one is called the *Foreign population*. A given cohort i , with $i = 1, \dots, m$, belonging to one of the two populations, is identified by the (common) initial age x_i at time zero. The set \mathcal{X} of initial ages is common to the two populations.

Domestic Population

The mortality intensity of each cohort x_i , for $i = 1, \dots, m$, belonging to the Domestic population is denoted with $\lambda_{x_i}^d$, or simply λ_i^d , and follows a non-mean reverting CIR process:

$$d\lambda_{x_i}^d(t) = (a_i + b_i\lambda_{x_i}^d(t))dt + \sigma_i\sqrt{\lambda_{x_i}^d(t)}dW_i(t), \quad (13)$$

where $a_i, b_i, \sigma_i, \lambda_{x_i}^d(0) \in \mathbb{R}^{++}$ are strictly positive real constants and the W_i 's are instantaneously correlated standard Brownian Motions, i.e. $dW_i(t)dW_j(t) = \rho_{ij}dt$ with $i, j \in \{1, \dots, m\}$. This implies that the mortality intensities of two different cohorts belonging to the Domestic Population are instantaneously correlated.

Foreign Population

The mortality intensity of cohort x_i belonging to the Foreign population is denoted with $\lambda_{x_i}^f$, or simply with λ_i^f , and is given by the convex combination of the mortality intensity of the corresponding cohort belonging to the Domestic population $\lambda_{x_i}^d$ and an idiosyncratic component λ' which affects the Foreign population only and that depends on the initial age x_i in a deterministic way, i.e.

$$\lambda_{x_i}^f(t) = \delta_i\lambda_{x_i}^d(t) + (1 - \delta_i)\lambda'(t; x_i), \quad (14)$$

where

$$d\lambda'(t; x_i) = (a(x_i; a') + b(x_i; b'))d\lambda'(t; x_i)dt + \sigma(x_i; \sigma', \gamma')\sqrt{d\lambda'(t; x_i)}dW'(t), \quad (15)$$

with $\delta_i \in [0, 1]$. The functions $a(x_i; a')$, $b(x_i; b')$, and $\sigma(x_i; \sigma', \gamma')$ are deterministic functions of the initial age x_i and, for every x_i with $i = 1, \dots, N$, they depend on the parameters $a' > 0$, $b' > 0$, $\sigma' > 0$, $\gamma' > 0$ respectively. W' is a standard Brownian Motion, that is assumed to be independent of W_i for each $i = 1, \dots, N$.

Intuitively, the idiosyncratic risk source W' is population specific in the sense that it is common to all the cohorts of the Foreign population. Two remarks are important.

- The idiosyncratic component $\lambda'(t; x_i)$ affects differently each cohort. The different impact on each cohort depends deterministically on their initial age x_i through the functional forms of $a(x_i; a')$, $b(x_i; b')$, and $\sigma(x_i; \sigma', \gamma')$;
- Each cohort x_i has, in general, a specific sensitivity to $\lambda'(t; x_i)$, that is given by the parameter δ_i .

Thus, the mortality intensities of two different cohorts of the Foreign population are correlated, because so are the corresponding cohorts of the Domestic population. In general, the correlation between $\lambda_{x_i}^f$ and $\lambda_{x_j}^f$ is different from the correlation between $\lambda_{x_i}^d$ and $\lambda_{x_j}^d$ because of the weights δ_i and δ_j . Moreover, we also model the non-perfect correlation between cohorts across the two populations, because of the presence of the idiosyncratic component λ' affecting the Foreign population.

The functional forms of $a(x_i; a')$, $b(x_i; b')$, and $\sigma(x_i; \sigma', \gamma')$ should be chosen in order to capture cohort effects in the Foreign Population. A possible choice is:

$$\begin{aligned} a(x_i; a') &= a'x_i, \\ b(x_i; b') &= b', \\ \sigma(x_i; \sigma') &= \sigma'e^{\gamma'x_i}, \end{aligned}$$

which means that the drift of $\lambda(t; x_i)$ is linearly increasing with x_i and its diffusion coefficient is instead exponentially increasing with x_i . However, the simplest specification of the model, which we adopt in our application, is:

$$\begin{aligned} a(x_i; a') &= a', \\ b(x_i; b') &= b', \\ \sigma(x_i; \sigma') &= \sigma'. \end{aligned}$$

3.1 Similarity/Diversification index

Building up on the longevity model described in the previous section, we propose a synthetic measure to describe the similarity/dissimilarity between the annuity portfolios written on two populations, that we define as *Similarity* index. Let n_i^d be the number of annuities written on cohort x_i belonging to the domestic population, n_i^f the number of annuities written on cohort x_i belonging to the foreign population, $n_i = n_i^d + n_i^f$ and m^f the number of generations in the foreign portfolio with non-zero n_i^f . Then the Similarity Index (SI) is equal to:

$$SI = 1 - \frac{1}{m^f} \sum_{i=1}^m \left[1 - \frac{n_i^d + n_i^f \delta_i}{n_i} \right]. \quad (16)$$

If $\delta_i = 1$ for every i , which means that the two portfolios are written on the same population, then, obviously, $SI = 1$. On the other hand, if $\delta_i = 0$ for every i , and $n_i^f \rightarrow \infty$ while n_i^d remains constant, we have that $SI \rightarrow 0$. We then define the Diversification index (*DI*) as the complement to 1 of the Similarity Index, interpreting it as a measure of the dissimilarity of the two portfolios. The higher the DI, the higher the diversification benefit that we should be expecting by coupling two portfolios.

4 Application

In this section, we try to quantify the diversification gains in an annuity portfolio in which UK is the Domestic country and Italy is the foreign one. We consider portfolios composed by 11 different cohorts: $x_i = 65, \dots, 75$.

4.1 Mortality intensities estimation

We calibrate the parameters of the mortality model to the generations of UK and Italian males whose age, at 31/12/2012, is between 64 and 74, that is, the cohorts born between 1937 and 1947. We use the 1-year \times 1-year cohort death rates data provided by the Human Mortality Database. The estimation of the parameters is

performed minimizing the RMSE between the observed and the model-implied survival probabilities using, for each cohort, the 20 observations from 1993 and 2012². Tables 1 and 2 report the calibrated parameters for the two populations, while Figures 1 and 2 report the actual and fitted survival probabilities and the calibration errors, respectively.

Table 1. Domestic Population (UK) calibration results.

Age	a	b	σ	λ_0	RMSE
65	$2.7878 \cdot 10^{-5}$	0.0723	0.0075	0.0116	0.00035
66	$6.5423 \cdot 10^{-5}$	0.0652	0.0059	0.0124	0.00028
67	$1.8424 \cdot 10^{-5}$	0.0740	0.0080	0.0135	0.00035
68	$5.3144 \cdot 10^{-5}$	0.0685	0.0084	0.0160	0.00043
69	$1.2500 \cdot 10^{-4}$	0.0589	0.0091	0.0164	0.00039
70	$8.4734 \cdot 10^{-5}$	0.0646	0.0108	0.0189	0.00056
71	$7.1323 \cdot 10^{-5}$	0.0667	0.0106	0.0212	0.00038
72	$4.1759 \cdot 10^{-5}$	0.0688	0.0073	0.0239	0.00040
73	$2.2984 \cdot 10^{-5}$	0.0689	0.0066	0.0262	0.00063
74	$9.6036 \cdot 10^{-5}$	0.0663	0.0131	0.0282	0.00040
75	$3.3898 \cdot 10^{-5}$	0.0684	0.0077	0.0316	0.00049

Table 2. Foreign Population (IT) calibration results.

Age	a'	b'	σ'	δ	RMSE	λ'_0
65				0.8071	0.00060	0.0075
66				0.8036	0.00073	0.0127
67				0.9348	0.00031	0.0190
68				0.8074	0.00045	0.0115
69				0.7893	0.00120	0.0163
70	$5.8458 \cdot 10^{-5}$	$4.2841 \cdot 10^{-11}$	$1.1464 \cdot 10^{-7}$	0.8119	0.00053	0.0141
71				0.7903	0.00099	0.0124
72				0.8006	0.00039	0.0092
73				0.8106	0.00064	0.0115
74				0.7622	0.00160	0.0209
75				0.8470	0.00053	0.0182

4.2 Correlation matrix estimation

To be able to estimate the instantaneous correlations implied by our longevity risk model, we apply the Gaussian Mapping technique, which we describe in the next section.

²These correspond to the last 20 observations for the Italian males. However, since the UK dataset is updated until 31/12/2013, we have excluded the last available observation for the UK cohorts.

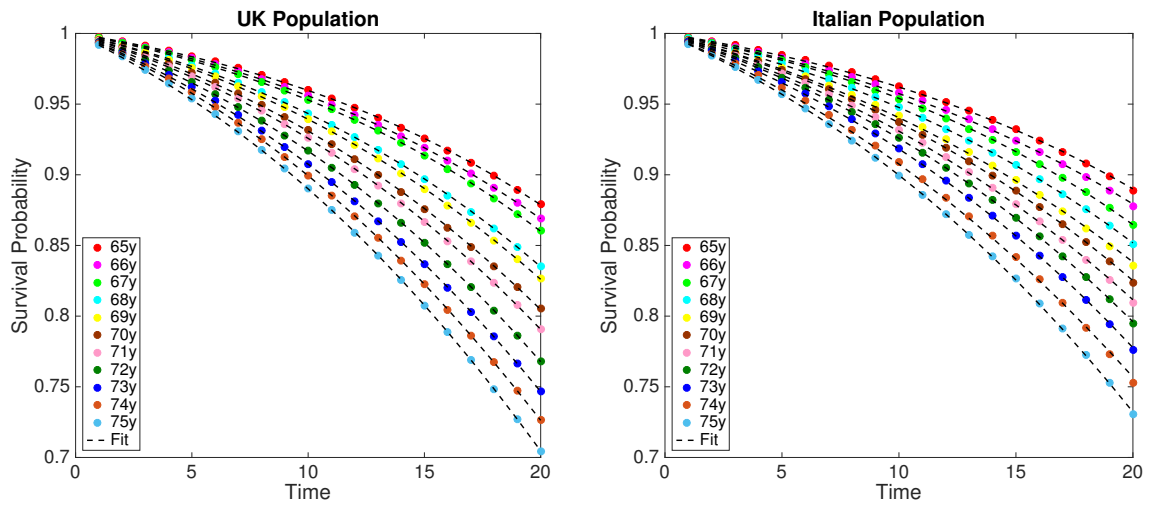


Figure 1. Observed and theoretical survival probabilities. The left panel shows the observed vs. fitted survival probabilities for the Foreign population, while the right reports the figures for the Domestic population.

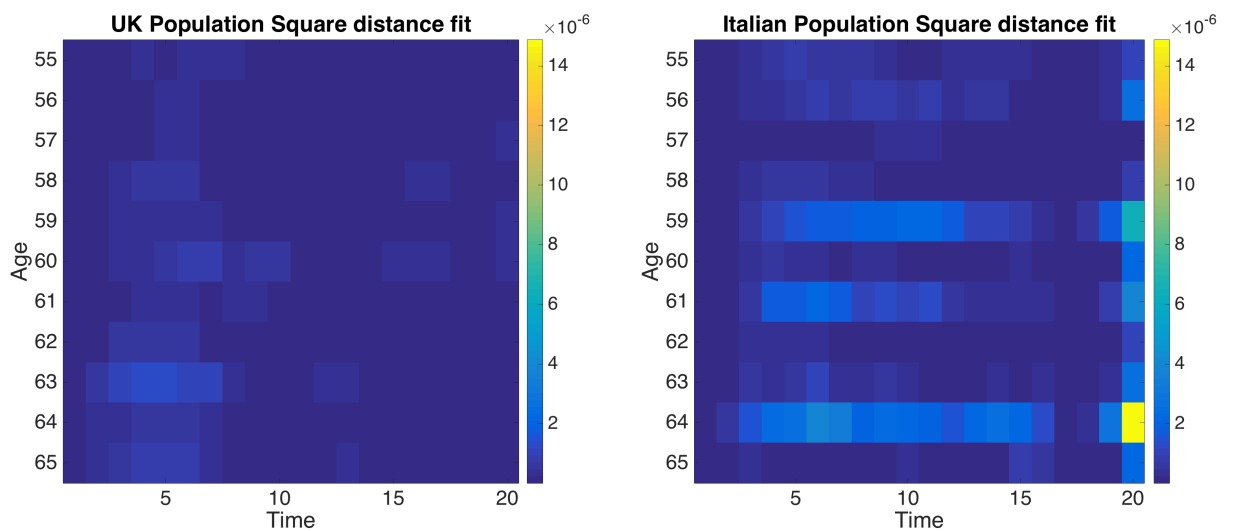


Figure 2. Calibration errors.

4.2.1 Gaussian Mapping

The Gaussian Mapping technique has been used in CDS pricing (see Brigo and Mercurio (2001)). It consists in mapping a CIR process into a Vasicek process that is as *close* as possible to the original one. While the meaning of *close* will be explained in what follows, we clarify that our objective is to use the mapped Vasicek processes to compute analytically the correlations between each λ_i^d and λ_j^d , with $i, j = 1, \dots, N$. Starting from the CIR process (13) describing the mortality intensity of cohort x_i belonging to the domestic population, we consider a Vasicek process driven by the same Brownian Motion $W_i(t)$, having the same drift and the same initial point:

$$d\lambda_i^V(t) = (a_i + b_i\lambda_{x_i}^V(t))dt + \sigma_i^V dW_i(t), \quad \lambda_i^V(0) = \lambda_{x_i}^d(0). \quad (17)$$

The instantaneous volatility coefficient σ_i^V of (17) is then determined by making the two processes are as close as possible. Here, by close we mean that the two processes agree on the survival probability for a fixed maturity T :

$$S_i^d(t, T) = S_i^V(t, T; \sigma_i^V). \quad (18)$$

Then, we approximate the correlation between $\lambda_i^d(t)$ and $\lambda_j^d(t)$ by the correlation between $\lambda_i^V(t)$ and $\lambda_j^V(t)$:

$$Corr_0(\lambda_i^d(t), \lambda_j^d(t)) \approx Corr_0(\lambda_i^V(t), \lambda_j^V(t)), \quad (19)$$

since this last correlation can be computed analytically. A simple application of Itô's Lemma allows us to show that the solution to the SDE (17) is given by:

$$\lambda_i^V(t) = \lambda_i^V(0)e^{b_it} + \frac{a_i}{b_i}(1 - e^{b_it}) + \sigma_i^V \int_0^t e^{b_i(t-s)} dW_i(s). \quad (20)$$

Therefore, we have that:

$$\mathbb{E}_0[\lambda_i^V(t)] = \lambda_i^V(0)e^{b_it} + \frac{a_i}{b_i}(1 - e^{b_it}) \quad (21)$$

$$Var_0[\lambda_i^V(t)] = \frac{(\sigma_i^V)^2}{2b_i} \sqrt{e^{2b_it} - 1}. \quad (22)$$

Since $\lambda_i^V(t) - \mathbb{E}_0[\lambda_i^V(t)] = \sigma_i^V \int_0^t e^{b_i(t-s)} dW_i(s)$, the covariance between $\lambda_i^V(t)$ and $\lambda_j^V(t)$ is:

$$\begin{aligned} Cov_0(\lambda_i^V(t), \lambda_j^V(t)) &= \mathbb{E}_0 \left[\sigma_i^V \sigma_j^V \left(\int_0^t e^{b_i(t-s)} dW_i(s) \right) \left(\int_0^t e^{b_j(t-s)} dW_j(s) \right) \right] \\ &= \mathbb{E}_0 \left[\sigma_i^V \sigma_j^V \rho_{ij} \int_0^t e^{(b_i+b_j)(t-s)} ds \right] \\ &= \sigma_i^V \sigma_j^V \rho_{ij} \int_0^t e^{(b_i+b_j)(t-s)} ds \\ &= \frac{\sigma_i^V \sigma_j^V \rho_{ij}}{b_i + b_j} \left(e^{(b_i+b_j)t} - 1 \right). \end{aligned}$$

Finally, we have:

$$\begin{aligned} \text{Corr}_0(\lambda_i^V(t), \lambda_j^V(t)) &= \frac{\text{Cov}_0(\lambda_i^V(t), \lambda_j^V(t))}{\sqrt{\text{Var}_0[\lambda_i^V(t)] \text{Var}_0[\lambda_j^V(t)]}} \\ &= \frac{2\rho_{ij}}{b_i + b_j} \cdot \frac{e^{(b_i+b_j)t} - 1}{\sqrt{\frac{(e^{2b_it}-1)(e^{2b_jt}-1)}{b_ib_j}}} \end{aligned} \quad (23)$$

Thanks to the Gaussian Mapping technique we can also compute the conditional correlation between two generations belonging to two different populations. Considering $0 \leq u \leq t$, the conditional correlation between $\lambda_{x_i}^d(t)$ and $\lambda_{x_j}^f(t)$ is given by:

$$\text{Corr}_u[\lambda_{x_i}^d(t), \lambda_{x_j}^f(t)] = \delta_j \frac{\text{Cov}_u(\lambda_{x_i}^d(t), \lambda_{x_j}^d(t))}{\sqrt{\text{Var}_u(\lambda_{x_i}^d(t)) \cdot \text{Var}_u(\lambda_{x_j}^f(t))}}, \quad (24)$$

where $\text{Cov}_u(\lambda_{x_i}^d(t), \lambda_{x_j}^d(t))$ is computed using the Gaussian mapping technique, and

$$\text{Var}_u(\lambda_{x_j}^f(t)) = \delta_j^2 \text{Var}_u(\lambda_{x_j}^d(t)) + (1 - \delta_j)^2 \text{Var}_u(\lambda'(t; x_j)). \quad (25)$$

4.2.2 Instantaneous correlation estimates

Using the central mortality rates data available in the UK life tables³, we estimate the instantaneous correlations ρ_{ij} between $d\lambda_i$ and $d\lambda_j$ by inverting the approximated correlation expression (23). To compute the correlations between our 11 cohorts, we start from the central mortality rates in 1958 of the people aged between 1 and 11, and we follow the diagonal of the life table until we reach the central mortality rates of the people aged between 65 and 75 in 2012. The central mortality rates table constructed this way has dimension 55×11 and allows to estimate the correlation coefficients which we report in Table 3. Correlations are close to 1, but they tend to decrease with the distance between the initial ages of the two considered cohorts. Table 4 shows the correlations across the two populations, while Figure 3 shows their covariance. The two generations with lowest covariance are 66 years old UK males and 66 years old Italian males.

Table 3. Instantaneous correlation matrix UK population. Colored cells highlight the minimum of each column.

	65	66	67	68	69	70	71	72	73	74	75
65	1										
66	0.9990	1									
67	0.9983	0.9992	1								
68	0.9983	0.9988	0.9989	1							
69	0.9973	0.9985	0.9988	0.9993	1						
70	0.9969	0.9979	0.9983	0.9989	0.9995	1					
71	0.9972	0.9977	0.9983	0.9987	0.9988	0.9986	1				
72	0.9964	0.9970	0.9977	0.9984	0.9987	0.9986	0.9994	1			
73	0.9962	0.9970	0.9976	0.9985	0.9988	0.9989	0.9992	0.9997	1		
74	0.9959	0.9967	0.9974	0.9983	0.9989	0.9991	0.9991	0.9995	0.9996	1	
75	0.9957	0.9960	0.9964	0.9974	0.9978	0.9981	0.9990	0.9996	0.9994	0.9995	1

³Source: Human Mortality Database.

Table 4. Correlation between populations. Rows are UK generations, columns are Italian generations. Colored cells highlight the minimum of each row.

	65	66	67	68	69	70	71	72	73	74	75
65	0.9821	0.9815	0.9803	0.9807	0.9803	0.9797	0.9798	0.9790	0.9789	0.9786	0.9785
66	0.9815	0.9830	0.9817	0.9817	0.9820	0.9812	0.9809	0.9800	0.9801	0.9799	0.9793
67	0.9803	0.9817	0.9819	0.9812	0.9817	0.9810	0.9809	0.9802	0.9802	0.9801	0.9795
68	0.9807	0.9817	0.9812	0.9827	0.9826	0.9820	0.9816	0.9813	0.9815	0.9813	0.9805
69	0.9803	0.9820	0.9817	0.9826	0.9839	0.9831	0.9823	0.9822	0.9824	0.9825	0.9814
70	0.9797	0.9812	0.9810	0.9820	0.9831	0.9834	0.9819	0.9819	0.9822	0.9823	0.9815
71	0.9798	0.9809	0.9809	0.9816	0.9823	0.9819	0.9831	0.9825	0.9824	0.9823	0.9823
72	0.9790	0.9800	0.9802	0.9813	0.9822	0.9819	0.9825	0.9830	0.9828	0.9826	0.9827
73	0.9789	0.9801	0.9802	0.9815	0.9824	0.9822	0.9824	0.9828	0.9832	0.9829	0.9826
74	0.9786	0.9799	0.9801	0.9813	0.9825	0.9823	0.9823	0.9826	0.9829	0.9832	0.9827
75	0.9785	0.9793	0.9795	0.9805	0.9814	0.9815	0.9823	0.9827	0.9826	0.9827	0.9833

4.3 Evaluating the diversification gains in terms of risk margin

In this section, we describe our simulation assumptions and present the evaluation of the geographical diversification effect. We consider a simulation time horizon of 30 years, a constant interest rate of 2% and we disregard the modelling of interest rate risk or exchange rates to focus on the impact of longevity risk. This choice is made to isolate and capture any possible added benefit specifically due to the geographic diversification of an annuity portfolio. The time horizon at which the Risk Margin is computed is 15 years. Consistently with the Solvency II regulation, we select a 99.5% confidence interval when calculating the Risk Margin associated to the portfolio.

Initial Portfolio

We consider a UK Insurer with an initial portfolio Π^0 , composed of contracts sold to males whose age, at 31/12/2012, is between 65 and 75. 100 contracts are sold to each generation and, therefore, the initial portfolio is composed of 1100 contracts. The initial Actuarial Value $AV_{\Pi^0}(0)$ of the portfolio is:

$$AV_{\Pi^0}(0) = 1.5288 \cdot 10^4, \quad (26)$$

while the Risk Margin computed at time 0 with a time horizon of 15 years and confidence interval $\alpha = 99.5\%$ is

$$RM_{\Pi^0}(0) = 1.3018 \cdot 10^3. \quad (27)$$

Hence, the initial portfolio value is

$$\Pi^0(0) = AV_{\Pi^0}(0) + RM_{\Pi^0}(0) = 1.6590 \cdot 10^4. \quad (28)$$

We observe that, in this case, the Risk Margin accounts for 8.52% of the initial portfolio Actuarial Value. The Risk Margin represents the amount of money that the Insurer needs to set aside in order to ensure its solvency (with a probability of 99.5%) in case of an unexpected increase in longevity over a 15 year period.

Domestic Expansion

With a Domestic Expansion, the Insurer doubles the size of its annuity portfolio,

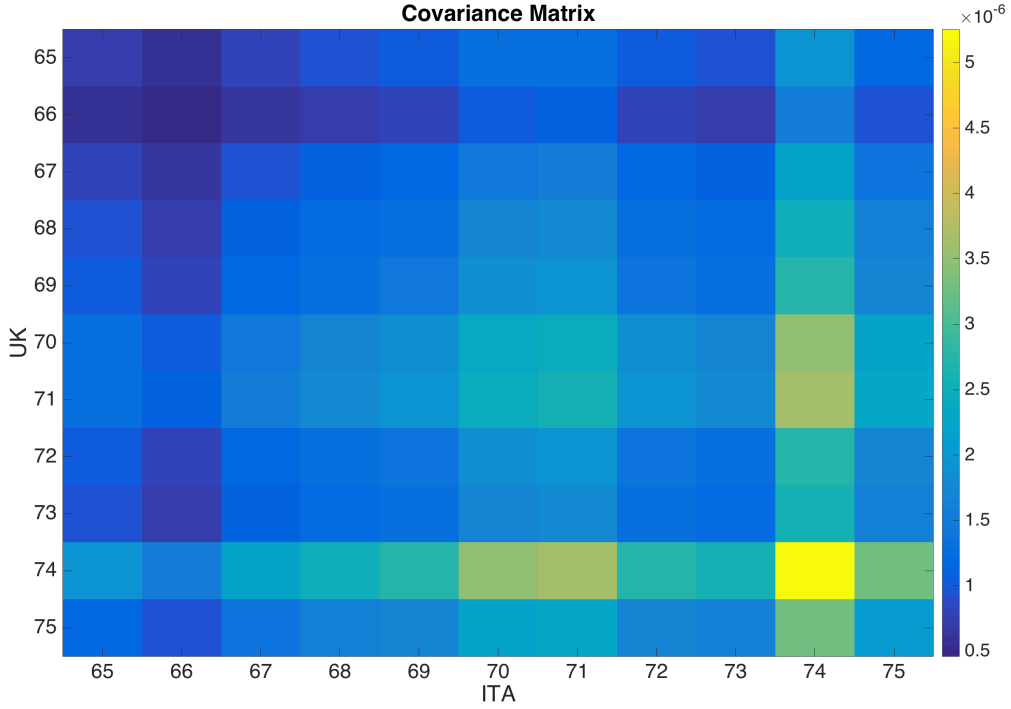


Figure 3. Covariance matrix between Italian and UK generations.

selling additional policies to her domestic population, i.e. the UK population. The new portfolio Π^1 is, therefore, composed of 2200 contracts, 200 for each generation. Hence,

$$AV_{\Pi^1}(0) = 3.0576 \cdot 10^4, \quad (29)$$

$$RM_{\Pi^1}(0) = 2.6036 \cdot 10^3, \quad (30)$$

$$\Pi^1(0) = 3.3179 \cdot 10^4. \quad (31)$$

The Risk Margin proportion relative to actuarial value is unaffected by the size of the portfolio, and still accounts for 8.52% of the Actuarial Value of the Domestically Expanded portfolio. In this case, we obviously have that the diversification index between Π^0 and $\Pi^1 - \Pi^0$ is 0, as no diversification gains can be obtained.

Foreign Expansion

In the case of a Foreign Expansion, the Insurer doubles the number of policies in its annuity portfolio by selling contracts written on policyholders belonging to the Foreign population, namely the Italian one. The new portfolio Π^2 is, therefore, composed of 1100 contracts sold to the UK population (100 for each generation) and of 1100 contracts sold to the Italian population (100 for each generation). Hence,

$$AV_{\Pi^2}(0) = 3.1252 \cdot 10^4, \quad (32)$$

$$RM_{\Pi^2}(0) = 2.4602 \cdot 10^3, \quad (33)$$

$$\Pi^2(0) = 3.3712 \cdot 10^4. \quad (34)$$

In this case, the Risk Margin accounts for 7.87% of the Actuarial Value of the Domestically Expanded portfolio. Now we have that the diversification index between Π^0 and $\Pi^2 - \Pi^0$ is 0.0925. Obviously, the size and the composition of the Foreign Portfolio affect the diversification gains. We then compute the values and risk margins of three additional portfolios, obtained by altering the number of policies sold to each different cohorts in the two populations. Table 5 summarizes the results, reporting the actuarial values, risk margins and total values of the portfolio. The column $\%RM$ represents the Risk Margin, expressed as a percentage of the Actuarial Value of the portfolio. Portfolio Π^3 is obtained fixing the number of policies sold to each generation of foreign policyholders to twice the number of policies in the initial domestic portfolio. Portfolio Π^F is exposed only to the foreign population, with 100 contracts sold for each generation, useful for comparison. The portfolio Π_{opt}^1 is obtained, like Π^1 , selling additional policies to the Domestic population. However, its composition is optimized to obtain the minimum risk margin achievable, under the constraint that the number of the new contracts is 1100. It can then be considered as the maximally diversified portfolio, in the absence of geographical diversification. Its percentage risk margin is 6.65% and is obtained by adding 1100 annuities sold to the UK 66 years old. Similarly, Π_{opt}^2 is obtained allowing for geographical diversification and optimizing the composition of the Foreign, Italian portfolio. The risk margin is 6.17%, obtained by selling the whole 1100 contracts to the Italian males aged 66. The DI of this last portfolio is the highest among the portfolios analyzed, 0.1801.

Table 5. Effects of geographical diversification ($r = 2\%$)

Portfolio	AV	RM	Π	$\%RM$	DI
Π^0	$1.5288 \cdot 10^4$	$1.3018 \cdot 10^3$	$1.6590 \cdot 10^4$	8.52%	-
Π^F	$1.5964 \cdot 10^4$	$1.1584 \cdot 10^3$	$1.7123 \cdot 10^4$	7.26%	-
Π^1	$3.0576 \cdot 10^4$	$2.6036 \cdot 10^3$	$3.3179 \cdot 10^4$	8.52%	0
Π^2	$3.1252 \cdot 10^4$	$2.4602 \cdot 10^3$	$3.3712 \cdot 10^4$	7.87%	0.0925
Π^3	$4.7217 \cdot 10^4$	$3.6186 \cdot 10^3$	$5.0835 \cdot 10^4$	7.66%	0.1233
Π_{opt}^1	$3.2979 \cdot 10^4$	$2.1947 \cdot 10^3$	$3.5173 \cdot 10^4$	6.65%	0
Π_{opt}^2	$3.3447 \cdot 10^4$	$2.0621 \cdot 10^3$	$3.5509 \cdot 10^4$	6.17%	0.1801

For the sake of completeness, Table 6 reports the results under the assumption of zero interest rate, i.e. $r = 0\%$.

Table 6. Effects of geographical diversification ($r = 0\%$)

Portfolio	AV	RM	Π	$\%RM$	DI
Π^0	$1.9097 \cdot 10^4$	$2.1318 \cdot 10^3$	$2.1228 \cdot 10^4$	11.16%	-
Π^F	$2.0093 \cdot 10^4$	$1.9060 \cdot 10^3$	$2.1999 \cdot 10^4$	9.49%	-
Π^1	$3.8193 \cdot 10^4$	$4.2636 \cdot 10^3$	$4.2457 \cdot 10^4$	11.16%	0
Π^2	$3.9189 \cdot 10^4$	$4.0378 \cdot 10^3$	$4.3227 \cdot 10^4$	10.30%	0.0925
Π^3	$5.9282 \cdot 10^4$	$5.9437 \cdot 10^3$	$6.5226 \cdot 10^4$	10.03%	0.1233
Π_{opt}^1	$4.1675 \cdot 10^4$	$3.6480 \cdot 10^3$	$4.5323 \cdot 10^4$	8.75%	0
Π_{opt}^2	$4.2400 \cdot 10^4$	$3.4234 \cdot 10^3$	$4.5824 \cdot 10^4$	8.07%	0.1801

Under this lower interest rate level, the magnitude of longevity risk is more severe, as expected: the percentage Risk Margins are higher for all portfolios, increasing in the best-case scenario to 8.07%, up from 6.17%.

5 Longevity Swap synthetic expansion

A foreign portfolio expansion like Π_{opt}^2 is in reality very difficult to achieve, because it is unlikely that an insurer can target the sale of annuity contracts only to a specific cohort. Moreover, if feasible, the expansion Π_{opt}^2 is likely to entail some cost $C_0 > 0$ that may reflect any cost related to entering a new market. Therefore, the real value of the liabilities of the insurer following the optimal foreign portfolio expansion can be rewritten as:

$$\bar{\Pi}_{opt}^2 = \Pi_{opt}^2 + C_0. \quad (35)$$

On the other hand, the same level of geographical diversification could be synthetically obtained through a longevity swap. Consider our UK life insurer with portfolio Π^0 , and assume that she sells a longevity swap written on 1100 individuals aged 66 belonging to the Italian population. Being the seller of the swap, the UK insurer will receive every year, until the maturity of the contract, a fixed quantity equal to K and will pay a stochastic amount given by the realized survival rate of the Italian 66 years old males. We also assume the maturity of the swap to be $T = \omega$. From the point of view of the seller, assuming independence between mortality and interest rate risk, the value at time t of the longevity swap is:

$$\begin{aligned} L(t, T) &= 1100 \sum_{\mathcal{T}=t+1}^{T-t} [K - S_{66}(t, \mathcal{T})] D(t, \mathcal{T}) = \\ &= 1100 \sum_{\mathcal{T}=t+1}^{T-t} \mathbb{E}_t \left[K - \exp \left(- \int_t^{\mathcal{T}} \lambda_{66}(s) ds \right) \right] \mathbb{E}_t \left[\exp \left(- \int_t^{\mathcal{T}} r(u) du \right) \right], \end{aligned} \quad (36)$$

where K is the swap rate, $S_{66}(t, \mathcal{T})$ is the (t, \mathcal{T}) -Survival probability for a 66 years old Italian male and $D(t, \mathcal{T})$ is the discount factor. Under no arbitrage, the swap rate is chosen in such a way that the value of the contract is zero at inception, that is:

$$K = \frac{\sum_{\mathcal{T}=1}^{T-t} \mathbb{E} \left[\exp \left(- \int_0^{\mathcal{T}} \lambda_{66}(s) ds \right) \right] \mathbb{E} \left[\exp \left(- \int_0^{\mathcal{T}} r(u) du \right) \right]}{\sum_{\mathcal{T}=1}^{T-t} \mathbb{E} \left[\exp \left(- \int_0^{\mathcal{T}} r(u) du \right) \right]}.$$

K is the value at time zero of an annuity contract sold to 66 year old Italian males divided by the time zero value of a non stochastic series of cash flows of one unit of currency, each year, until the maturity of the contract. In our case, assuming a constant interest rate of 2%, we have that $K = 0.7218$.

Intuitively, equation (36) can be spitted in two parts:

$$+ 1100K \sum_{\mathcal{T}=t+1}^{T-t} D(t, \mathcal{T}), \quad (37)$$

and

$$- 1100 \sum_{\mathcal{T}=t+1}^{T-t} S_{66}(t, \mathcal{T}) D(t, \mathcal{T}). \quad (38)$$

The second term represents the liabilities obligations of the UK insurer deriving from the longevity swap. $\tilde{\Pi}_{opt}^2$ denotes the value of the liability portfolio of the UK insurer given by Π^0 plus the longevity swap. The actuarial value of $\tilde{\Pi}_{opt}^2$ is then given by

$$AV_{\tilde{\Pi}_{opt}^2}(t) = AV_{\Pi^0}(t) + 1100 N_{66}^{IT}(t) = AV_{\Pi_{opt}^2}(t), \quad (39)$$

where N_{66}^{IT} is the value of an annuity contract sold to a 66 year old Italian male. Obviously, the risk margin of $\tilde{\Pi}_{opt}^2$ is:

$$RM_{\tilde{\Pi}_{opt}^2}(t) = D(t, t+T) \cdot VaR_{\alpha}(AV_{\tilde{\Pi}_{opt}^2}(t+T) - \mathbb{E}_t[AV_{\tilde{\Pi}_{opt}^2}(t+T)]) = \quad (40)$$

$$= D(t, t+T) \cdot VaR_{\alpha}(AV_{\Pi_{opt}^2}(t+T) - \mathbb{E}_t[AV_{\Pi_{opt}^2}(t+T)]) \quad (41)$$

$$= RM_{\Pi_{opt}^2}(t). \quad (42)$$

We have indeed just shown that the longevity swap allows the insurer to achieve the same actuarial value and the same risk margin of a direct sale of annuity contracts to the Italian males. Also in this case, however, the sales of the longevity swap may entail some initial cost C'_0 given, for instance, by the required due diligence actions. Hence, the value of the liability portfolio $\tilde{\Pi}_{opt}^2$ is given by:

$$\tilde{\Pi}_{opt}^2 = \Pi_{opt}^2 + C'_0. \quad (43)$$

As long as $C'_0 < C_0$, the UK insurer will find in the synthetic expansion through the longevity swap a more attractive solution.

6 Conclusions

In this paper, we discussed the possible benefits of geographically diversified portfolios, due to non-perfect correlation between populations's longevity risks. We have considered the problem of an insurer who has to decide whether to expand his liability portfolio in the country where it is based or in a foreign country. We pointed out that some diversification gains may be realized when expanding internationally, due to the mitigation of its exposure to longevity risk. To discuss whether these gains may be sizable in an annuity portfolio, we first proposed a longevity risk model that, while being parsimonious, can be calibrated so as to capture the non-perfect correlations among the different cohorts of two different populations. We computed the risk margin coherently with the Solvency II modelling approach, as a loading on the actuarial value of the portfolio, computed as a 99.5% VaR of the whole portfolio value. Our application, based on an annuity portfolio written on the UK and the Italian populations, shows that a non-negligible reduction of the portfolio risk margins can be obtained by expanding internationally. Under a 0% interest rate assumption, we show that an optimally chosen foreign portfolio, on top of the domestic one, can lower the

risk margin by more than 2% as a percentage of the actuarial value of the portfolio. This example can be considered as a conservative one, since the two male populations of UK and Italy we have analyzed present rather similar historical mortality dynamics. The diversification effect is shown to be more relevant, the lower the risk-free interest rate. We also introduce the alternative option of a synthetic expansion of the portfolio of the insurer, performed taking a short position on a longevity swap having the foreign population as underlying. Such synthetic expansion can be theoretically designed to replicate the diversification benefit obtained via a physical expansion. The choice between the physical or synthetic expansions depends on their costs.

Our paper contributes to the literature on the modeling of longevity risk, by proposing a multi-population model that captures the dependence structure within and across populations. It contributes to the understanding of longevity risk management, by pointing out that, although the same increasing longevity trend is common to the populations of almost every country, geographical diversification may lead to some benefit. Our model is able to quantify such effects. Finally, even though we framed our problem as representing the decision of an insurer, analogous consideration can be made in the case of a reinsurer willing to provide insurance to portfolios of policies written on different populations. This can have implications for securitization and for the structuring of longevity derivative products.

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