

A Multi-Curve Random Field LIBOR Market Model

Abstract

A multi-curve random field LIBOR market model is proposed by extending the LIBOR market model (LMM) with uncertainty modelled by a random field to the multi-curve framework, where the forward LIBOR curve for projecting future cash flows and the curve for discounting are modelled distinctively but jointly. The multi-curve methodology is introduced recently in the literature to account for the increased basis among closely related interest rates since the 2007-2009 credit crisis. Closed-form formulas for pricing caplets and swaptions are derived. Then the multi-curve random field LIBOR market model is integrated with the local and stochastic volatility models (lognormal-mixture, SABR, Wu and Zhang (2006)) to capture the implied volatility skew/smile. Finally, we estimate various models from market data. Empirical results show that for both in-sample and out-of-sample pricing, the random field LIBOR Market Model outperforms the Brownian motion LIBOR Market Model. Moreover, their corresponding multi-curve variations outperform their single-curve counter-parts respectively.

Key Words: LIBOR market model, random field, multi-curve term structure model, credit crisis.

JEL Code: G12.

1 Introduction

After the credit crisis started in the summer of 2007, interest rates quoted in the market showed non-negligible inconsistency. Before the crisis, for example, LIBOR and OIS (Overnight Index Swap) rates for the same maturity tracked each other within a distance of a few basis points. Similarly FRA rates and the corresponding forward rates implied by consecutive deposit rates would be quoted at a negligible spread. However, the market forces of the crisis in 2007 widened these spreads. Figure 1 shows the historical series of the U.S. LIBOR Deposit 6-month (6M) rates versus the U.S. Overnight Indexed Swap (OIS) 6-month (6M) rates from March 15, 2005 to September 14, 2012 and Figure 2 reports the historical series of quoted USD Forward Rate Agreement (FRA) 3×6 rates versus the forward rates implied by the

corresponding USD OIS 3M and 6M rates. We can see that the basis was well below ten basis points until summer 2007, but it has since then widened significantly.

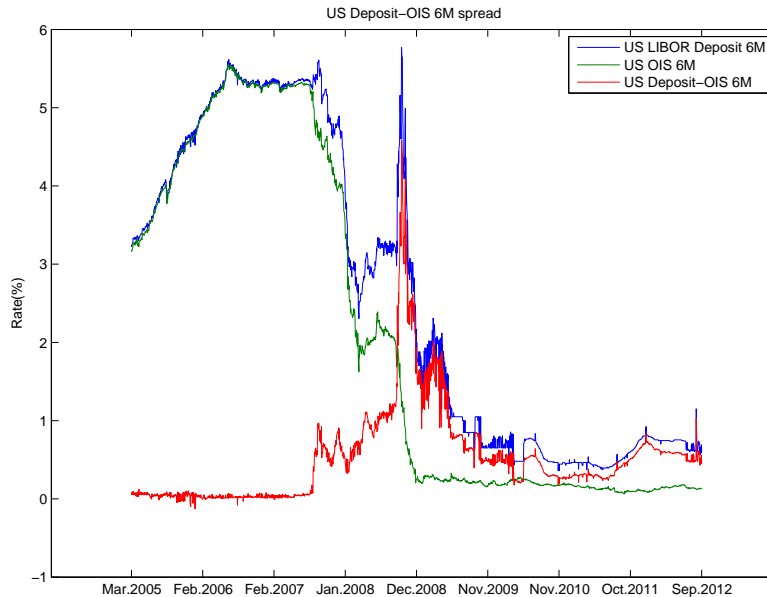


Figure 1: U.S. LIBOR Deposit-6M(spot) Rates vs U.S. OIS-6M Rates. Quotations Mar.15.2005-Sep.14.2012(source: Bloomberg)

A new theoretical framework has been proposed to deal with these phenomena. Morini [51] proposed a theoretical framework that explained the divergence of interest rates by introducing a stochastic default probability. However practitioners seemed to agree on an empirical approach, which was based on the construction of as many curves as possible tenors (e.g. 1-month, 3-month, 6-month). Future cash flows are thus generated through the curves associated with the underlying rates and then discounted by some other curve. This approach is termed the multi-curve method. Mercurio [52], Kijima et al. [46], Chibane et al. [18], Henrard [27], Ametrano and Bianchetti [7], Ametrano [5] and Fujii et al. [22, 23, 24] among others, have contributed to the development of this approach. In particular, Mercurio [52] derived the LIBOR market model(LMM) within multi-curve framework.

LIBOR market model (Brace, Gatarek, and Musiela (BGM) [15], Jamshidian [31], and Miltersen, Sandmann, and Sondermann [50]) is based on the assumption that each forward LIBOR rate follows a driftless Brownian motion under its own forward measure, which justifies the use of Black’s formula for the pricing of caplets and swaptions. LMM is very flexible in the choice of volatility and correlation structure and is easy to calibrate. Due to these

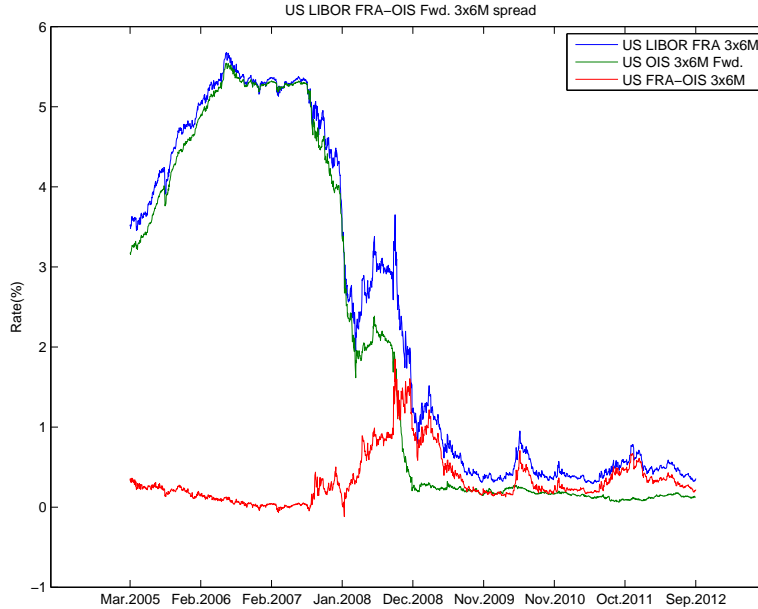


Figure 2: U.S. LIBOR FRA-3×6M Rates vs U.S. OIS-3×6M Fwd.Rates vs. Quotations Mar.15.2005-Sep.14.2012(source: Bloomberg)

desirable features, LMM has become very popular among practitioners for interest rate modeling and derivatives pricing.

In this paper, we extend Mercurio [52]’s work by describing the uncertainty terms as random fields. The benefits of random field models is that it is no longer necessary to determine the number of factors in advance. Random field models also do not need to be re-calibrated frequently. Random field modeling of interest rates is first introduced by Kennedy [41, 42] and Goldstein [25] and other authors like Pang [57], Longstaff et al.[48] and Bester [9] investigate different aspects of this methodology such like calibration and option pricing, etc. Limiting the scope to Gaussian random field, Kennedy [41] derives the no arbitrage condition on the drift of the instantaneous forward rate process similar to that in the HJM model. Goldstein [25] extends the work to the case of non-Gaussian random field. Wu and Xu (2014) combine the tractability of the LIBOR Market Model with the benefits of random field models by driving the uncertainty in LIBOR Market Models with a random field. In this paper, we extend the Random Field LIBOR Market Model in Wu and Xu (2014) to the multi-curve setting to account for the increased basis between the forward LIBOR curves projecting future cash flows and the risk-free discount curve. The multi-curve random field LMM is further integrated with the local/stochastic volatility models to capture implied volatility skews/smiles. As a result, this model

produces implied volatility smile and therefore can be calibrated more accurately to market data. Finally, we estimate various models from historical data using unscented Kalman Filter and examine both in-sample and out-of-sample pricing performance. Our empirical results show that the random field model outperforms the Brownian motion model. Moreover, multi-curve models outperform single-curve models for both.

The rest of the paper is organized as follows. Section 2 reviews the multi-curve pricing methodology and random field modeling. Section 3 extends the multi-curve LIBOR market model to a random field setting. Formulas for caplets and swaptions are provided. Section 3 integrates the model with log-mixture local volatility models. Section 4 estimates the models using historical market data and examines pricing accuracy. Section 5 concludes the paper.

2 Multi-curve Pricing Methodology and Random Field Models

We first introduce random field modeling. Define zero coupon bond price

$$P(t, T) = e^{-\int_t^T f(t, u) du}, \quad (1)$$

where $f(t, u)$ is the instantaneous forward rate and t, T are current time and maturity respectively. According to Goldstein [25], if we require the discounted bond price

$$e^{-\int_0^t r(s) ds} P(t, T), \quad (2)$$

where $r(t)$ is the instantaneous spot rate, to be a martingale under risk neutral measure \mathbb{Q} , the drift term must be given by $\sigma(t, T) \int_t^T \sigma(t, u) c(t, T, u) du$, to satisfy the no-arbitrage condition, where $c(t, T, u)$ is defined in Eq.(4). Thus the dynamics of the instantaneous forward rates $f(t, T)$ under risk neutral measure is given as

$$df(t, T) = \sigma(t, T) \int_t^T \sigma(s, u) c(s, T, u) du ds + \sigma(t, T) d\widetilde{W}(t, T), \quad (3)$$

with the correlation structure

$$\text{Corr}[dW(t, T_1), dW(t, T_2)] = c(t, T_1, T_2), \quad (4)$$

where $\lim_{\Delta T \rightarrow 0} c(t, T, T + \Delta T) = 1$ and $\widetilde{W}(t, T)$ is a random field under risk neutral measure \mathbb{Q} .

Heath, Jarrow and Morton [28] proved the existence of risk neutral measure directly and built up a framework to pricing all contingent claims. Rather than identify the risk neutral measure, Goldstein [25] first assumes its existence and derives the dynamics of the instantaneous forward rates

under this measure, then shows the existence of the risk neutral measure within a general equilibrium framework.

Before we introduce the multi-curve pricing methodology, we first review the traditional single curve pricing approach. First, we select one set of suitable interest instruments in market and build a single yield curve using the preferred bootstrapping procedure. Second, we compute the forward rates, cash flows, discount factors on this curve, under an appropriate measure. Third, we price the derivatives by summing up the discounted cash flows and hedge the resulting delta risk. However, this approach is not consistent with market practice anymore after the credit crisis. As pointed out in Bianchetti [10], it does not take into account the fact that the interest rate market is segmented into sub-areas corresponding to instruments with distinct underlying rate tenors. This fact may explain the interest rates inconsistency observed in market. For example, the divergence of LIBOR deposit rates and OIS rates, the spreads of FRA and the corresponding forward rates implied by consecutive deposits, and the basis swap spreads.

In practice the multiple curve approach prevailed on the market. This approach is based on the construction of two different kinds of yield curves, the discounting curve and forwarding curves. The single discounting curve is used to calculate discount factors and thus cash flow's present values. The multiple forwarding curves, built from market instruments corresponding to the underlying tenor, are used to calculate future cash flows based on forward rates with the corresponding rate tenor. With this approach, interest rate derivatives with a given rate tenor should be priced and hedged using the instruments with the same underlying rate tenor. Ametrano and Bianchetti [7] and Bianchetti [10] summarized the multiple curve pricing methodology after credit crisis in the following procedure.

- Build one discounting curve ζ_d using the preferred choice of interest rate instruments and bootstrapping procedure.
- Build multiple forwarding curves $\zeta_{f_1}, \dots, \zeta_{f_n}$ using preferred choice of distinct sets of interest rate instruments and bootstrapping procedure, with the same underlying LIBOR rate tenor for the corresponding curves.
- Compute the forward rates $F^{f_i}(t)$ with tenor τ_i using the corresponding forwarding curve ζ_{f_i} .
- Compute the relevant discount factor $P(t, T)$ from the discounting curve ζ_d .
- Compute the price of the derivative at time t as the sum of the discounted cash flow $\sum_{k=1}^m P(t, T_k) E^{T_k} [cf(F^{f_i}(t))]$, where $cf(F^{f_i}(T))$ is the payoffs and the expectation is taken with respect to T_k -forward measure, associated with discount numeraire $P(t, T_k)$.

We use this methodology for our multiple curve pricing in this paper. We can construct i forwarding curves if the derivatives depend on the forward rates with i different tenors. In this case we can assume that there are two kinds of curves, curve ζ_d for discounting with associated discount factor $P(t, T)$ and curves ζ_{f_i} for projecting future cash flows. The forward rates can be defined for the two different kinds of curves with associated discount factors. The construction of the forward curve ζ_{f_i} is similar as in the pre-credit-crunch situation except that only the market quoted instruments corresponding to the tensor are employed in the bootstrapping procedure. For example, the three-month (3M) forward curve can be constructed by zero coupon rates from the 3M deposit rates, the 3M FRA and the 3M swap rates. The discount curve can be selected differently depending on contract to be priced. For example, it can be OIS curve if there is no counterparty risk or deposit rates if there exists risk.

Mercurio [52] extended the LIBOR market model(LMM) consistently with the two-curve assumption, namely joint evolutions of FRA rates and LIBOR rates. In the single curve case, a FRA rate can be defined as the expectation of the corresponding LIBOR rate under a given forward measure. However, in Mercurio's multi-curve setting LIBOR rate and the forward measure belong to different curves. FRA rates are thus different from LIBOR rates and can be modeled with their own dynamics. Following Mercurio [52], to show how to value the interest rate derivatives in two-curve setting of distinct forward and discount curves, we consider an interest rate swap where the floating leg pays the LIBOR forward rates $L_k^{f_i}(t)$ from forward curve ζ_f . The time t value of the floating leg payoff is

$$\delta_k P(T, T_k) E^{T_k} [L_k^{f_i}(T_{k-1}) | \mathcal{F}_t], \quad (5)$$

where the expectation is under the forward measure associated with discount curve ζ_d . Define the time t forward rate agreement(FRA) rate $F_k^{f_i}(t)$ as the fixed rate to be exchanged at time T_k for floating payment $\delta_k L_k^{f_i}(T_{k-1})$ so that the swap rate has zero value at time t , i.e.

$$F_k^{f_i}(t) := E^{T_k} [L_k^{f_i}(T_{k-1}) | \mathcal{F}_t]. \quad (6)$$

The net present value of the swap's floating leg is given by summing the single payoffs

$$\sum_{k=i+1}^j \delta_k P(t, T_k) F_k^{f_i}(t). \quad (7)$$

Thus the swap rate is

$$S_{i,j}(t) = \frac{\sum_{k=i+1}^j \delta_k P(t, T_k) F_k^{f_i}(t)}{\sum_{k=i+1}^j \delta_k P(t, T_k)}. \quad (8)$$

Indeed, the multi-curve framework reduces to the single-curve framework when the discounting curve and forwarding curves coincide, i.e., $\zeta_{f_i} = \zeta_d$,

for all i . In the single-curve framework, $L_k(t)$ is a martingale under the associated T_k -forward measure \mathbb{Q}^{T_k} , thus we have that $F_k^{f_i}(t) = L_k^{f_i}(t)$, which makes Eq.(7) to become $P(t, T_i) - P(t, T_j)$. Hence Eq.(8) becomes

$$S_{i,j}(t) = \frac{P(t, T_i) - P(t, T_j)}{\sum_{k=i+1}^j \delta_k P(t, T_k)}, \quad (9)$$

which is exactly the formula of swap rate in single-curve framework.

3 Multi-curve Random Field LIBOR Market Model

In this section we derive the multi-curve random field LIBOR market model, as well as the multi-curve random field local or stochastic volatility models in the multi-curve framework. For simplicity we investigate the case that there is only one curve used to project future cash flows and the curve used for discounting is different from it. We follow the approach of Brace et al. [15] and Miltersen et al. [50] by assuming that forward LIBOR rates have lognormal diffusions and then extend the random field LIBOR market model to multi-curve framework in the spirit of Mercurio [52]. First, we derive the multi-curve random field LIBOR market model in Sec.3.1, as well as the closed-form formulas for pricing European caplets and swaptions in Sec.3.2. Second, random field local and stochastic volatility models in multi-curve framework are derived in Sec.3.3 and Sec.3.4, respectively.

3.1 Multi-curve Random Field LIBOR Market Model

In this section we derive the dynamics of LIBOR rates $L_k(t)$ from discount curve and FRA rates $F_k^f(t)$ from forward curve with uncertainty term modeled with random field under risk neutral measure \mathbb{Q} and T_j -forward measure \mathbb{Q}^{T_j} for $j = 1, \dots, N$.

Let us consider time structures $\{T_0, T_1, \dots, T_N\}$ with intervals $\delta_k = T_k - T_{k-1}$, $1 \leq k \leq N$. For $t < T_{k-1} < T_k$, in two-curve setting we need to model the evolution of FRA rates $F_k^f(t)$ from forward curve ζ_f under T_k -forward measure:

$$dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t, u) dB^{T_k}(t, u) du, \quad (10)$$

where $B^{T_k}(t, u)$ is a Brownian field with correlation

$$\text{corr}[dB(t, T_1), dB(t, T_2)] = c_f(T_1, T_2).$$

Suppose that the LIBOR forward rate $L_k(t)$ from the discount curve has dynamics

$$dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) dW^{T_k}(t, u) du, \quad (11)$$

where

$$\text{corr}[dW(t, T_1), dW(t, T_2)] = c_d(T_1, T_2).$$

In addition we also need to specify the correlation between the two random fields

$$\text{corr}[dB(t, T_1), dW(t, T_2)] = c_{df}(T_1, T_2).$$

By Brigo and Mercurio [14], for $k < j$, in two-curve case, the FRA rate $F_k^f(t)$ has the drift term

$$\begin{aligned} -dF_j^f(t) d \ln \frac{P(t, T_j)}{P(t, T_k)} &= -dF_j^f(t) \ln(1 / [\prod_{i=k+1}^j (1 + \delta_i L_i(t))]) \\ &= \sum_{i=k+1}^j \frac{\delta_i}{1 + \delta_i L_i(t)} \frac{dF_j^f(t) dL_i(t)}{dt} \\ &= F_j^f(t) \int_{T_{j-1}}^{T_j} \eta_j(t, u) \left[\sum_{i=k+1}^j \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, u) c_{df}(v, u)}{1 + \delta_i L_i(t)} dv \right] du. \end{aligned}$$

The derivation in case $j < k$ is analogous. Thus we have the following theorem.

Theorem 3.1. (Two-curve random field dynamics under forward measures) *Under the lognormal assumptions, the dynamics of the instantaneous forward rates $L_k(t)$ and FRA rates $F_k^f(t)$ under the T_j -forward measure, in three cases $j < k, j = k, j > k$, are described respectively by*

$$\left\{ \begin{array}{l} dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) [dW^{T_j}(t, u) + \Lambda_j^k(t, u) dt] du, \\ dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t, u) [dB^{T_j}(t, u) + \Lambda_j^k(t, u) dt] du, \quad j < k; \\ dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) dW^{T_j}(t, u) du, \\ dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t, u) dB^{T_j}(t, u) du, \quad j = k; \\ dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, u) [dW^{T_j}(t, u) + \Lambda_k^j(t, u) dt] du, \\ dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t, u) [dB^{T_j}(t, u) + \Lambda_k^j(t, u) dt] du, \quad j > k; \end{array} \right. \quad (12)$$

with

$$\Lambda_j^k(t, u) = \sum_{i=j+1}^k \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, v) c_d(v, u)}{\delta_i L_i(t) + 1} dv,$$

and

$$\Lambda_j^k(t, u) = \sum_{i=j+1}^k \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, v) c_{df}(v, u)}{\delta_i L_i(t) + 1} dv,$$

where $W^{T_j}(t, u)$ is a Gaussian random field under T_j -forward measure. The above equations admit a unique solution if the coefficient $\xi_k(\cdot, \cdot)$ are locally bounded, locally Lipschitz continuous and predictable.

3.2 Option Pricing in Multi-curve Random Field LIBOR Market Model

In this section we derive the closed-form Black implied volatility formulas for caplets and swaptions in multi-curve random field LIBOR market model, using two-curve setting as an example.

3.2.1 Multi-curve random field LMM formula for caplets

The payoff of a caplet at time T_k is

$$\delta_k [L_k^f(T_{k-1}) - K]^+.$$

The caplet price at time t becomes

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E^{T_k} [[L_k^f(T_{k-1}) - K]^+ | \mathcal{F}_t].$$

Since in two curve setting the pricing measure is the T_k -forward measure for discount curve, the LIBOR forward rate $L_k^f(t)$ from forward curve is not a martingale under the measure \mathbb{Q}^{T_k} . Mercurio [52] introduced a way to solve it. From the definition of FRA rate Eq.(6), we know that

$$F_k^f(T_{k-1}) = L_k^f(T_{k-1}),$$

which means that the price for caplet in two curve setting can be written as

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) E^{T_k} [[F_k^f(T_{k-1}) - K]^+ | \mathcal{F}_t]$$

Suppose that the FRA rate $F_k^f(t)$ has dynamics Eq.(12) under T_k -forward measure, then the price is given as

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \mathbf{Black}(K, F_k^f(t), \sigma_k^{Black,MR} \sqrt{T_{k-1} - t}),$$

where

$$\sigma_k^{Black,MR} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \left[\int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} \eta_k(s, x) \eta_k(s, y) c_f(x, y) dx dy \right] ds}. \quad (13)$$

3.2.2 Multi-curve random field LMM formula for swaptions

The payoff of a swaption at time T_k is

$$\delta_k [S_{i,j}(T_{k-1}) - K]^+.$$

The time t -price is therefore

$$[S_{i,j}(t) - K]^+ \sum_{k=i+1}^j \delta_k P(t, T_k),$$

where

$$\begin{aligned} S_{i,j}(t) &= \frac{\sum_{k=i+1}^j \delta_k P(t, T_k) F_k^f(t)}{\sum_{k=i+1}^j \delta_k P(t, T_i)} \\ &= \sum_{k=i+1}^j \delta_k \frac{P(t, T_k)/P(t, T_j)}{\sum_{k=i+1}^j \delta_k P(t, T_k)/P(t, T_j)} F_k^f(t) \\ &= \sum_{k=i+1}^j \frac{\delta_k \prod_{l=k+1}^j (1 + \delta_l L_l(t))}{\sum_{k=i+1}^j \delta_k \prod_{l=k+1}^j (1 + \delta_l L_j(t))} F_k^f(t) \end{aligned}$$

is the swap rate. Similarly to the discussion in Sec.3.2.1, we have

$$\ln S_{i,j}(t) = \ln \left[\sum_{k=i+1}^j \delta_k \prod_{l=k+1}^j (1 + \delta_l L_l(t)) F_k^f(t) \right] - \ln \left[\sum_{k=i+1}^j \delta_k \prod_{l=k+1}^j (1 + \delta_l L_j(t)) \right].$$

By Itô's formula, we know that

$$\frac{1}{S_{i,j}(t)} \frac{\partial S_{i,j}(t)}{\partial L_k(t)} = \frac{\alpha_k^{i,j}(t) \delta_k}{1 + \delta_k L_k(t)}, \quad \frac{1}{S_{i,j}(t)} \frac{\partial S_{i,j}(t)}{\partial F_k^f(t)} = \frac{\beta_k^{i,j}(t) \delta_k}{1 + \delta_k L_k(t)},$$

where

$$\begin{aligned} \alpha_k^{i,j}(t) &= \frac{\sum_{h=i+1}^{k-1} \delta_h F_h^f(t) \prod_{l=h+1}^j (1 + \delta_l L_l(t))}{\sum_{h=i+1}^j \delta_h F_h^f(t) \prod_{l=h+1}^j (1 + \delta_l L_l(t))} \frac{\sum_{h=i+1}^{k-1} \delta_h \prod_{l=h+1}^j (1 + \delta_l L_l(t))}{\sum_{h=i+1}^j \delta_h \prod_{l=h+1}^j (1 + \delta_l L_l(t))}, \\ \beta_k^{i,j}(t) &= \frac{\prod_{h=k}^j (1 + \delta_h L_h(t))}{\sum_{h=i+1}^j \delta_h F_h^f(t) \prod_{l=h+1}^j (1 + \delta_l L_l(t))}. \end{aligned}$$

Using Itô's formula for two variables, the uncertainty term of $dS_{i,j}(t)$ is given by

$$\sum_{k=i+1}^j \frac{1}{S_{i,j}(t)} \left[\frac{\partial S_{i,j}(t)}{\partial L_k} \int_{T_{k-1}}^{T_k} \xi_k(t, u) dW^{T_k}(t, u) + \frac{\partial S_{i,j}(t)}{\partial F_k^f} \int_{T_{k-1}}^{T_k} \eta_k(t, u) dB^{T_k}(t, u) \right],$$

or equivalently

$$\sum_{k=i+1}^j \int_{T_{k-1}}^{T_k} \left[\frac{\alpha_k^{i,j}(t) \delta_k L_k(t) \xi_k(t, u)}{1 + \delta_k L_k(t)} dW^{T_k}(t, u) + \frac{\beta_k^{i,j}(t) \delta_k L_k(t) \eta_k(t, u)}{1 + \delta_k L_k(t)} dB^{T_k}(t, u) \right] du.$$

Thus the Black implied volatility of $S_{i,j}(t)$ is defined as

$$\begin{aligned}
& \overline{\sigma}_{i,j}^{Black,MR}(t) \\
&= \sqrt{\frac{1}{T_i - t} \int_t^{T_i} \left\| \sum_{k=i+1}^j \int_{T_{k-1}}^{T_k} \left[\frac{\alpha_k^{i,j}(s) \delta_k L_k(s) \xi_k(s, u)}{1 + \delta_k L_k(s)} dW^{T_k}(s, u) \right. \right. \\
&\quad \left. \left. + \frac{\beta_k^{i,j}(s) \delta_k L_k(s) \eta_k(s, u)}{1 + \delta_k L_k(s)} dB^{T_k}(s, u) \right] du \right\|^2 ds} \\
&= \sqrt{\frac{1}{T_i - t} \int_t^{T_i} \left\| \sum_{k=i+1}^j \frac{\delta_k L_k(s)}{1 + \delta_k L_k(s)} \int_{T_{k-1}}^{T_k} [\alpha_k^{i,j}(s) \xi_k(s, u) dW^{T_k}(s, u) \right. \\
&\quad \left. + \beta_k^{i,j}(s) \eta_k(s, u) dB^{T_k}(s, u)] \right\|^2 ds} \\
&= \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^j \sum_{l=i+1}^j \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \left\{ \int_{T_{k-1}}^{T_k} [\alpha_k^{i,j}(s) \xi_k(s, u) \right. \\
&\quad \left. dW^{T_k}(s, u) + \beta_k^{i,j}(s) \eta_k(s, u) dB^{T_k}(s, u)] \times \int_{T_{l-1}}^{T_l} [\alpha_l^{i,j}(s) \xi_l(s, u) dW^{T_l}(s, u) \right. \\
&\quad \left. + \beta_l^{i,j}(s) \eta_l(s, u) dB^{T_l}(s, u)] \right\} ds} \\
&= \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^j \sum_{l=i+1}^j \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \left\{ \int_{T_{k-1}}^{T_k} \int_{T_{l-1}}^{T_l} [\alpha_k^{i,j}(s) \right. \\
&\quad \xi_k(s, u) \alpha_l^{i,j}(s) \xi_l(s, v) c_d(u, v) + \beta_k^{i,j}(s) \eta_k(s, u) \beta_l^{i,j}(s) \eta_l(s, v) c_f(u, v) \\
&\quad + \alpha_k^{i,j}(s) \xi_k(s, u) \beta_l^{i,j}(s) \eta_l(s, v) c_{df}(u, v) \\
&\quad \left. + \beta_k^{i,j}(s) \eta_k(s, u) \alpha_l^{i,j}(s) \xi_l(s, v) c_{df}(u, v)] dudv \right\} ds.} \tag{14}
\end{aligned}$$

The third equation is obtained by using standard freezing approximation techniques, i.e., approximately evaluating the LIBOR rates $L_k(s)$, $t \leq s \leq T_i$, appearing in the instantaneous volatility at initial time t .

The above equation is too complex. To simplify the formula we resort to a simple approximation technique. We know that the swap rate $S_{i,j}(t)$ can be written as a linear combination of FRA rates $F_k^f(t)$:

$$S_{i,j}(t) = \sum_{k=i+1}^j \omega_k^{i,j}(t) F_k^f(t),$$

with

$$\omega_k^{i,j}(t) = \frac{\delta_k P(t, T_k)}{\sum_{k=i+1}^j \delta_k P(t, T_i)}.$$

We freeze the weights $\omega_k^{i,j}(t)$ at their time t value and denote it by $\omega_k^{i,j}$.

Thus we have the approximation:

$$S_{i,j}(t) = \sum_{k=i+1}^j \omega_k^{i,j} F_k^f(t),$$

which leads to

$$dS_{i,j}(t) = \sum_{k=i+1}^j \omega_k \int_{T_{k-1}}^{T_k} \eta_k(t, u) F_k^f(t) dB_k^{T_k}(t, u) du.$$

Notice that by freezing the weights we are in fact freezing the dependence of $S_{i,j}(t)$ on $L_k(t)$. Thus the Black implied volatility is defined approximately as:

$$\begin{aligned} \bar{\sigma}_{i,j}^{Black,MR} &= \sqrt{\frac{1}{T_i - t} \int_t^{T_i} \left\| \sum_{k=i+1}^j F_k^f(s) \omega_k^{i,j} \int_{T_{k-1}}^{T_k} \eta_k(s, u) dB(s, u) \right\|^2 ds} \\ &= \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^j \sum_{l=i+1}^j F_k^f(t) \omega_k^{i,j} F_l^f(t) \omega_l^{i,j} \\ &\quad \times \int_t^{T_i} \int_{T_{k-1}}^{T_k} \int_{T_{l-1}}^{T_l} \eta_k(t, x) \eta_l(t, y) c_f(x, y) dx dy ds}. \end{aligned} \quad (15)$$

3.3 Multi-curve Random Field Local Volatility Models

In this section we derive the random field lognormal mixture model derived in multi-curve framework. Analogous to Eq.(10), we can assume that under T_k -forward measure, the dynamics of $L_k(t)$ and $F_k^f(t)$ with random field are given as

$$\begin{cases} dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW^{T_k}(t, u) du; \\ dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t, F_k^f(t), u) dB^{T_k}(t, u) du. \end{cases}$$

Let us consider the diffusion process with dynamics given by

$$dH_i^k(t) = H_i^k(t) \int_{T_{k-1}}^{T_k} \mu_i^k(t, H_i^k(t), u) dB^{T_k}(t, u) du \quad (16)$$

with initial value $H_i^k(0) = F_k^f(0)$ for all $i = 1, 2, \dots, M$, where $B^{T_k}(t, u)$ is a Gaussian field under T_k -forward measure with correlation $c_f(T_1, T_2)$. Analogous to Brigo and Mercurio [14], the problem here is to derive the local volatility $\eta_k(t, u)$ such that the density of $F_k^f(t)$ under T_k -forward measure satisfies, for each time t :

$$q_i^k(x, t) = \frac{d}{dx} P^{T_k} \{F_k^f(t) \leq x\} = \sum_{i=1}^M \omega_i \frac{d}{dx} P^{T_k} \{H_i^k(t) \leq x\} du = \sum_{i=1}^M \omega_i q_i^k(x, t),$$

where ω_i is a weight function with $\sum_{i=1}^M \omega_i = 1$. In fact $q_i^k(x)$ is a proper density function under T_k -forward measure since

$$\int_0^{+\infty} x q_k(x, t) dx = \int_0^{+\infty} x \sum_{i=1}^M \omega_i q_i^k(x, t) dx = \sum_{i=1}^M \omega_i H_i^k(0) = F_k^f(0).$$

The last calculation comes from the fact that $H_i^k(t)$ is a martingale under T_k -forward measure. We know that the local volatility $\eta_k(t, F_k^f(t))$ is

$$\iint_{T_{k-1}}^{T_k} \eta_k(t, u) \eta_k(t, v) c_f(t, u, v) dv du = \frac{\sum_{i=1}^M \omega_i [\iint_{T_{k-1}}^{T_k} \mu_i^k(t, u) \mu_i^k(t, v) c_f(t, u, v) dv du] q_i^k(x, t)}{\sum_{i=1}^N \omega_i q_i^k(x, t)}.$$

If we take $c_f(t, x, y) = 1$, the above formula reduces to

$$\eta_k(t) = \sqrt{\frac{\sum_{i=1}^M \omega_i \mu_i^k(t)^2 q_i^k(x, t)}{\sum_{i=1}^N \omega_i q_i^k(x, t)}}, \quad (17)$$

which is the original lognormal-mixture model derived in Brigo and Mercurio [14].

Since

$$\int_{T_{k-1}}^{T_k} \eta_k(t, F_k(t), u) dB^{T_k}(t, u) du$$

has normal distribution with variance

$$\iint_{T_{k-1}}^{T_k} \eta_k(t, F_k(t), u) \eta_k(t, F_k(t), x) c_d(t, u, v) dv du,$$

we have the following theorem:

Theorem 3.2. (Random field lognormal-mixture model dynamics under forward measures) *The dynamics of FRA rate $F_k^f(t)$ is given by*

$$dF_k^f(t) = F_k^f(t) \sqrt{\frac{\sum_{i=1}^M \omega_i (\iint_{T_{k-1}}^{T_k} \mu_i^k(t, u) \mu_i^k(t, v) c_f(t, u, v) dv du) q_i^k(x, t)}{\sum_{i=1}^M \omega_i q_i^k(x, t)}} dB^{T_k}(t), \quad (18)$$

where $W^{T_k}(t)$ is a Brownian motion under T_k -forward measure.

If we assume that in Eq.(16)

$$\mu_i^k(t, x, u) = \mu_i^k(t, u), \quad (19)$$

i.e. the densities $p_i^k(x, t)$ are all lognormal, where for all k , $\mu_i^k(t)$ are deterministic and continuous functions of time that are bounded from above and below by strictly positive constants, then the marginal density of $G_i^k(t)$ is lognormal and given by

$$q_i^k(x, t) = \frac{1}{xU_i^k(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2U_i^k(t)^2} \left[\ln \frac{x}{F_k(0)} + \frac{1}{2}F_i^k(t)^2\right]^2\right\}; \quad (20)$$

$$U_i^k(t) = \sqrt{\int_0^t \iint_{T_{k-1}}^{T_k} \mu_i^k(s, u)\mu_i^k(s, v)c_d(s, u, v)dudvds}. \quad (21)$$

3.3.1 Option pricing in random field local volatility models

In this section, we will derive the closed-form pricing formulas of caplets in two-curve random field lognormal mixture model.

The payoff of a caplet at time T_k is

$$\delta_k[L_k(T_{k-1}) - K]^+.$$

Following the discussion in Sec.3.2.1, the caplet price at time t becomes

$$\begin{aligned} \mathbf{Cplt}(t, K, T_{k-1}, T_k) &= \delta_k P(t, T_k) E^{T_k} [[L_k^f(T_{k-1}) - K]^+ | \mathcal{F}_t] \\ &= \delta_k P(t, T_k) E^{T_k} [[F_k^f(T_{k-1}) - K]^+ | \mathcal{F}_t] \\ &= P(0, T_k) \int_0^{+\infty} [x - K]^+ q_k(x, t) dx \\ &= P(0, T_k) \sum_{i=1}^M \int_0^{+\infty} [x - K]^+ q_i^k(x, t) dx. \end{aligned} \quad (22)$$

Suppose that the FRA rate $F_k^f(t)$ has dynamics as Eq.(16) under T_k -forward measure. Then the caplet price is given as

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \sum_{i=1}^M \mathbf{Black}(K, F_k(t), U_i^k(T_{k-1}, u)).$$

Given the above analytical tractability, we can derive an explicit approximation for the caplet implied volatility as a function of the caplet strike.

Theorem 3.3. (Implied volatility of random field lognormal-mixture model) Define $m = \ln \frac{F_k(t)}{K}$. The implied volatility $\sigma_k^{Black, MRF}$ is

$$\begin{aligned} &\sigma_k^{Black, MRF}(m) \\ &= \sigma_k^{Black, MRF}(0) + \frac{1}{2\sigma_k^{Black, MRF}(0)(T_{k-1} - t)} \sum_{i=1}^M \omega_i \left[\frac{\sigma_k^{Black, MRF}(0) \sqrt{T_{k-1} - t}}{V_i^k(T_{k-1})} \right. \\ &\quad \left. e^{\frac{1}{8}(\sigma_k^{Black, MRF}(0))^2(T_{k-1} - t) - U_i^k(T_{k-1})^2} - 1 \right] m^2 + o(m^2), \end{aligned}$$

where the at-the-money implied volatility $\sigma_k^{Black,MRF}(0)$ is given by

$$\sigma_k^{Black,MRF}(0) = \frac{2}{\sqrt{T_{k-1} - t}} \Phi^{-1} \left(\sum_{i=1}^M \omega_i \Phi \left(\frac{1}{2} U_i^k(T_{k-1}) \right) \right). \quad (23)$$

3.4 Multi-curve Random Field Stochastic Volatility Models

In this section we derive the approximate solution for caplets and swaptions in the case of stochastic volatility. We take the stochastic volatility structures as an example, one by Hagan et al.[39] (SABR) and one by Wu and Zhang [77].

3.4.1 SABR approach

It is natural to model the FRA rate in two curves case as

$$\begin{cases} dF_k^f(t) = F_k^f(t)^{\beta'_k} \int_{T_{k-1}}^{T_k} V'_k(t, u) dB^{T_k}(t, u) du, \\ dV'(t, u) = \alpha'_k(t) V'_k(t, u) dB^{T_k}(t, u), \end{cases}$$

with correlation structure

$$corr[dB(t, T_1), dB(t, T_2)] = c_f(T_1, T_2),$$

where $\beta'_k > 1, \alpha'$ are positive constants. Analogous to previous works, the dynamics of $F_k^f(t)$ and $V'_k(t, u)$ under the T_j -forward measure, in three cases $j < k, j = k, j > k$, are described respectively by

$$\begin{cases} \begin{cases} dF_k^f(t) = F_k^f(t)^{\beta'_k} \int_{T_{k-1}}^{T_k} V'_k(t, u) [dB^{T_j}(t, u) + \Psi_j^k(t, u) dt] du, \\ dV'_k(t, u) = V'_k(t, u) \alpha'_k(t, u) [dB^{T_j}(t, u) + \Psi_j^k(t, u) dt], \end{cases} & j < k; \\ \begin{cases} dF_k^f(t) = F_k^f(t)^{\beta'_k} \int_{T_{k-1}}^{T_k} V'(t, u) dB^{T_j}(t, u) du, \\ dV_k(t, u) = V_k(t, u) \alpha'_k(t, u) dB^{T_j}(t, u), \end{cases} & j = k; \\ \begin{cases} dF_k^f(t) = F_k^f(t)^{\beta'_k} \int_{T_{k-1}}^{T_k} V'_k(t, u) [dB^{T_j}(t, u) + \Psi_k^j(t, u) dt] du, \\ dV'_k(t, u) = V'_k(t, u) \alpha'_k(t, u) [dB^{T_j}(t, u) + \Psi_k^j(t, u) dt], \end{cases} & j > k; \end{cases} \quad (24)$$

with

$$\Psi_k^j(t, u) = \sum_{i=k+1}^j \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t)^{\beta_k} V_i(t, v) c_{df}(t, v, u)}{\delta_i L_i(t) + 1} dv.$$

The model above does not have analytic closed-form solutions and we can present a version of stochastic Taylor expansion in the spirit of Milstein scheme in random field setting.

3.4.2 Wu and Zhang (2006) approach

It is natural to model the FRA rate in two curves case as

$$\begin{cases} dF_k^f(t) = \sigma'_k F_k^f(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} dB^{T_k}(t, u) du, \\ dV'(t, u) = \kappa'(\theta' - V'(t, u))dt + \epsilon' \sqrt{V'(t, u)} dB^T(t, u), \end{cases}$$

with

$$\text{corr}(dB(t, u), dB(t, v)) = c_f(t, u, v),$$

where $\sigma'_k, \kappa', \theta', \epsilon'$ are positive constants. Since the measure change does not change the correlation of Brownian motion, the above correlation can exist under both measures, \mathbb{Q}^{T_k} and \mathbb{Q}^T . The dynamics of $F_k^f(t)$ and $V'_k(t, u)$ under the T_j -forward measure, in three cases $j < k, j = k, j > k$, are described respectively by

$$\begin{cases} dF_k^f(t) = \sigma_k F_k^f(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} [dB^{T_j}(t, u) + \Upsilon'_{j+1}(t, u) dt] du, \\ dV'(t, u) = \kappa'(\theta' - V'(t, u))dt - \epsilon' \sqrt{V'(t, u)} [dB^{T_j}(t, u) + \Upsilon'_{\bar{x}}(t, u) dt], & j < k; \\ dF_k^f(t) = \sigma_k F_k^f(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} dB^{T_k}(t, u) du \\ dV'(t, u) = \kappa'(\theta' - V'(t, u))dt - \epsilon' \sqrt{V'(t, u)} [dB^{T_j}(t, u) + \Upsilon'_{\bar{x}}(t, u) dt], & j = k; \\ dF_k^f(t) = \sigma_k F_k^f(t) \int_{T_{k-1}}^{T_k} \sqrt{V'(t, u)} [dB^{T_j}(t, u) + \Upsilon'_{k+1}(t, u) dt] du, \\ dV'(t, u) = \kappa'(\theta' - V'(t, u))dt - \epsilon' \sqrt{V'(t, u)} [dB^{T_j}(t, u) + \Upsilon'_{\bar{x}}(t, u) dt], & j > k; \end{cases} \quad (25)$$

with

$$\Upsilon'_{\bar{x}}(t, u) = \sum_{i=\bar{x}(t)}^j \int_{T_{i-1}}^{T_i} \frac{\delta_i \sigma_i L_i(t) \sqrt{V(t, u)} c_{df}(t, v, u)}{1 + \delta_i L_i(t)} dv.$$

The model above does not have analytic closed-form solutions and we can present a version of stochastic Taylor expansion in the spirit of Milstein scheme in random field setting.

In this section, we have derived the random field volatility models. We first derived the dynamics of LIBOR forward rate $L_k(t)$ for random field lognormal mixture model in Eq.(12), and the implied volatility formula in Eq.(18), which can be used to capture the caplet volatility smile on the calibration procedure. Then we derived the dynamics of LIBOR forward rate $L_k(t)$ for multi-curve random field stochastic volatility models, in Eq.(25) for SABR type and in Eq.(25) for Wu-Zhang type respectively. Although there are no closed-form option pricing formulas for random field stochastic volatility models, we can use the stochastic Taylor expansion to derive the discrete versions of dynamics, which can be used to price options numerically.

4 Empirical Implementation

The market models we discuss and derive in this paper require in general three different inputs, the initial curve, the instantaneous volatility, and the correlation structure. The ways to get the inputs are discussed as follows. 1)The initial yield curve and the corresponding forward rates can be bootstrapped from market zero-coupon bond prices. 2)The instantaneous volatilities of forward rates are usually assumed to depend on current time t , forward rate maturity T and time to maturity $T - t$. The most popular assumption is that the volatilities depend only on time to maturity, which is called time homogenous. There are two main options in choosing the volatility function. One is the piecewise-constant form where volatility is assumed to be constant between each time step. Another option is to assume some parametric forms for the volatilities. In this paper we assume that the instantaneous volatility has a particular parametric forms with several time independent parameters. 3)The estimation of correlation structures can be based on historically estimated correlation matrix or some parametric forms with desired features. However, the historical matrix is very volatile and is likely to have some outliers caused by bad estimation data. Thus the best idea is to assume a parametric functional form which has the desired structure features. See Sec.4.1 for details of the specification of model inputs.

4.1 Model specification

In this paper, the goal of LIBOR rate models calibration is to estimate the instantaneous volatility functions $\xi_k(t)$, $k = 1, 2, \dots, N$ and the correlation matrix $\rho_{i,j}(t)$, $i, j = 1, 2, \dots, N$ from the data of the initial yield curve $L_k(0)$ and the caps and swaptions prices observed on the market. Notice that in the random field case, the correlation structure may take the continuous form $c(u, v)$. In this paper, we use parametric forms to describe the instantaneous volatility $\xi_k(t)$ and the correlation structure $c(u, v)$.

The desired qualitative features of the instantaneous volatility usually come from the term structure of volatility which is directly observable from market. Volatilities generally depend on three factors: current time t , maturity T and time to maturity $\tau = T - t$. By Rebonato [59] the forward rate volatilities can also be deterministic functions of the full history of yield curve and its stochastic drivers or some stochastic quantities whose future values are known only in a statistical sense. Moreover, the instantaneous volatility could itself be a diffusion process. The most popular assumption is the time-homogeneity of the volatilities. Rebonato [60] proposed a time homogenous linear-exponential functional form with four parameters $\xi_k(t) = f(t, T_k)h(t)g(T_k)$ for the instantaneous volatility, while $h(t), g(T_k)$ are usually taken to be 1 and

$$f(t, T_k) = [a + b(T_k - t)]e^{-c(T_k - t)} + d; a, b, c, d > 0. \quad (26)$$

Thus $\xi_k(t) = f(t, T_k)$. By Rebonato [60], this formulation is reasonable since it satisfies criteria of a desired volatility function. First, the function is flexible enough to be able to produce either a humped or monotonically decreasing instantaneous volatility. Second, the function parameters have clear econometric interpretation to allow sanity check of calibration. For example, the parameter d is the volatility with large number of $\delta = T - t$, while the amount $a + d$ is approximately the instantaneous volatility of the forward rate with small δ . And the extreme of the function is reached at $(b - ca)/cb$. Third, analytical integration of the functions square should be possible to allow fast calculation of forward rate variance and covariance.

The correlation structure differs for LMM and random field LMM. The LMM needs the specification of the functional form of correlation matrix $\rho_{i,j}$, while the random field LMM needs the correlation functions $c(u, v)$. The difference of correlation structures for LMM and random field LMM is discussed as follows.

For LMM, we know that the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\frac{\text{Cov}(dL_i(t), dL_j(t))}{\sqrt{\text{Var}(dL_i(t))\text{Var}(dL_j(t))}} = \frac{\xi_i(t)\xi_j(t)dW_i(t)dW_j(t)}{\sqrt{\xi_i^2(t)\xi_j^2(t)}} = dW_i(t)dW_j(t) = \rho_{i,j}(t),$$

which means that the instantaneous correlation of forward rates is exactly the same as the correlation structure of Brownian motion $W(t)$. Thus the correlation matrix $\rho(t)$ should have the desirable features of the forward rates correlation structures, which is the historical correlation matrix of forward rates quoted in market. Meanwhile, the historical correlation matrix has two main features. First, the correlation matrix should always be decreasing when moving away from diagonal, which corresponds to the fact that forward rates with maturities close to each other tend to be more likely to move to the same direction as the rates with distinct maturities. Second, the correlation matrix should be increasing along sub-diagonal. We know that yield curve tends to flatten and forward rates with long maturities generally move in a more correlated way than rates with short maturities. Thus the parametric correlation matrix should fulfill such two features.

The classical function form to describe correlation is

$$\rho_{i,j} = \rho_0 + (1 - \rho_0)e^{-\rho_\infty|i-j|}, \quad (27)$$

where ρ_0 and ρ_∞ are positive. However, the problem of classical form is that it is not increasing along the sub-diagonal. To solve this problem, many authors created different forms of parametric correlation matrix. For example, Schoenmakers and Coffey [63] proposed a two-parameter function form to describe the correlations:

$$\rho_{i,j} = e^{-\frac{|i-j|}{N-1}(\rho_\infty + \rho_0 \frac{N-i-j+1}{N-2})}, \quad (28)$$

where N is the size of the matrix ρ and $0 \leq \rho_0 \leq \rho_\infty$. The correlation structure Eq.(28) can capture the two features of the correlation of forward rates and we use Eq.(28) as our parametric form for correlation matrix in calibration. The number of factors in LIBOR market model depends on the rank of correlation matrix ρ . A rank- d correlation matrix ρ with entries defined in Eq.(28) gives rise to a d -factor LIBOR market model. In this paper we take a full rank correlation matrix and thus the LIBOR market model considered in this paper has the number of factors the same as the number of forward rates considered. Notice that if we specify the correlation directly, we need to estimate about $K(K+1)/2$ parameters in estimation procedure, where K is the size of the correlation matrix. The number of parameters will become very large if K is large. Thus the techniques of factor reduction will be apply to reduce the rank of the matrix and thus the number of parameters. We use functional form to describe the correlation structure. The parameters needed to be estimated are just the parameters from the functional form. Thus it is not necessary to reduce the correlation matrix to low-rank.

However, for random field LMM, the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\begin{aligned} & \frac{Cov(dL_i(t), dL_j(t))}{\sqrt{Var(dL_i(t))Var(dL_j(t))}} \\ = & \frac{\int_{T_{i-1}}^{T_i} \int_{T_{j-1}}^{T_j} \xi_i(t, x) \xi_j(t, y) c(x, y) dx dy}{\sqrt{\int \int_{T_{i-1}}^{T_i} \xi_i(t, x) \xi_i(t, y) c(x, y) dx dy \int \int_{T_{j-1}}^{T_j} \xi_j(t, x) \xi_j(t, y) c(x, y) dx dy}}, \end{aligned}$$

which means that the instantaneous correlation of forward rates depends on both the correlation structure and the instantaneous volatilities. In non-random field case, the correlation matrix need to satisfy two features, decreasing when moving away from diagonal and increasing across sub-diagonal. However, in random field case, we can see that the two features of correlation matrix can be explained by the specification of $\xi(t, y)$ and $c(x, y)$, but not $c(x, y)$ only. Thus the choice of correlation functional form $c(x, y)$ for random field LIBOR market model can be simpler than that for LIBOR market model and it turns out that simple function form is enough. It will save much time on calibration. This is also a significant advantage of random field model. Analogous to Eq.(27), we only need to take

$$c(x, y) := e^{-\rho_\infty |x-y|}. \quad (29)$$

The correlation is independent of current time t .

As discussed in Sec.2, the spirit of multi-curve modeling is that interest rate derivatives should be priced using instruments with the same underlying rate tenor. In the HJM multi-curve framework, there are one discounting

curve and many forward curves, one for each quoted LIBOR rate tenor. The problem of building different curves corresponding to different rate tenors has been addressed in the work of some pioneers. For example, Bianchetti [10] treats the different curves as if they are curves for different currencies. Pallavicini and Tarengi [56] bootstrap the yield curves by interpolating on the spreads between modified forward rates of different tenors, along with their spread with respect to forward calculated from discount curve. In this thesis we follow the approach of Ametrano and Bianchetti [7], where they solve the problem in terms of different yield curves coherent with basic derivatives.

Since the swaptions in our data set are based on swaps against 6 month LIBOR, we only construct the 6 month tenor forward curve. Following Ametrano and Bianchetti [7], in multi-curve framework, the discount curve is obtained from U.S. Overnight Indexed Swaps and the forward rates to project the floating leg of swap cash flows are bootstrapped from 6 month FRA rates up to two years, swaps from two to thirty years paying an annual fixed rate in exchange for 6 month LIBOR, using the OIS curve as discounting curve. By Pallavicini and Tarengi [56], the choice of discounting curve using OIS rates can be justified by the fact that interbank operations are usually collateralized and it is straightforward to use overnight rate for discounting if we assume that the collateral is revalued daily. On the other hand, in the single curve framework, the yield curve used for both discounting cash flows and projecting swap floating leg payments are obtained as usual, short term LIBOR deposits(blew 1 year), mid-term FRA on LIBOR 3M (below 2 years)and mid/long-term swaps on LIBOR 6M(after 2 years). The details of strapping procedure can be found in Ametrano and Bianchetti [7] and Bianchetti et al. [13]. We can build the curves as follows.

1. LIBOR standard curve: the classic yield curve bootstrapped from short term LIBOR deposits(blew 1 year), mid-term FRA on LIBOR 3M (below 2 years)and mid/long-term swaps on LIBOR 6M(after 2 years). In single-curve modeling this curve will be used for discounting and forwarding curve are bootstrapped using as discounting curve.
2. OIS curve: the curve bootstrapped from the U.S. OIS rates. In two-curve modeling this curve will be used as discount curve.
3. LIBOR 6M curve: the LIBOR-OIS 6M curve bootstrapped from the LIBOR deposit 6M, mid-term FRA on LIBOR-6M(up to 2 years)and mid/long-term swaps on LIBOR 6M(after 2 years). In two curve modeling this curve will be used as forward curve.

In this section, we provide the closed-form Black implied volatility for caplets and swaptions, in the form of the instantaneous volatility $\xi(t, T)$ and the correlation structure $c(u, v)$. While extended to multi-curve framework, the model also needs the same inputs for other rates which are modeled

simultaneously with LIBOR rates. Thus in multi-curve framework the number of inputs may increase according to the number of curves used and the stochastic volatility models may be more complicated.

We may need to specify parametric forms for LIBOR rate $L_k(t)$ and FRA rates $F_k^f(t)$, the instantaneous volatility functions $\xi_k(t)$, $\eta_k(t)$ and the correlation structure $c_d(T_i, T_j)$, $c_f(T_i, T_j)$, $c_{df}(T_i, T_j)$. The functional form of the instantaneous volatility should satisfy the criteria of desired volatility. As shown in Sec.4.1, we can take $\xi_k(t, T_k) = f(t, T_k)h(t)g(T_k)$ and $\eta_k(t, T_k) = e(t, T_k)h(t)g(T_k)$, while for lognormal mixture model we take $\nu_i^k(t, T_k) = f_i(t, T_k)h(t)g(T_k)$. $h(t)$, $g(T_k)$ are equal to 1 and $f(t, T_k)$ is given as Eq.(26). The choice of correlation functional forms for two-curve random field LIBOR market model is the same as that for random field LIBOR market model. This means that it also shares the advantages of random field model. Analogous to Eq.(27), we only need to take

$$c_d(x, y) = e^{-\rho_{\infty, d}|x-y|} := c_d(x, y); \quad (30)$$

$$c_f(x, y) = e^{-\rho_{\infty, f}|x-y|} := c_f(x, y); \quad (31)$$

$$c_{df}(x, y) = e^{-\rho_{\infty, df}|x-y|} := c_{df}(x, y). \quad (32)$$

The correlation is independent of current time t .

The Black implied volatilities for caplets and swaptions are derived as follows. For LMM, we have that

$$\sigma_k^{Black}(t) = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} f^2(s, T_k) ds}, \quad (33)$$

and

$$\begin{aligned} \bar{\sigma}_{i,j}^{Black}(t) &= \sqrt{\frac{1}{T_i - t} \sum_{l,k=i+1}^j \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k F_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \rho_{kl}(s) f(s, T_k) f(s, T_l) ds} \\ &= \sqrt{\frac{1}{T_i - t} \sum_{l,k=i+1}^j \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k F_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \sigma_k^{Black} \sigma_l^{Black} \Theta_{l,k}(t) \rho_{k,l}}, \quad (34) \end{aligned}$$

where

$$\Theta_{l,k}(t) = \frac{\sqrt{T_l - t} \sqrt{T_k - t} \int_t^{T_i} f(s, T_k) f(s, T_l) ds}{\sqrt{\int_t^{T_k} f^2(s, T_k) ds} \sqrt{\int_t^{T_l} f^2(s, T_l) ds}}. \quad (35)$$

For random field LMM, we have that

$$\sigma_k^{Black, RF} = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \left[\int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} f(s, x) f(s, y) c(x, y) dx dy \right] ds}.$$

To allow for comparison of Brownian motion and random field LMM, it is reasonable to set $f(t, x) = f(t, T_k)$ for $x \in [T_{k-1}, T_k]$. Thus the above equation becomes

$$\begin{aligned}\sigma_k^{Black,RF} &= \sqrt{\frac{1}{T_{k-1}-t} \int_t^{T_{k-1}} \delta_k^2 f^2(s, T_k) ds \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c(x, y) dx dy} \\ &= \sqrt{\frac{1}{T_{k-1}-t} c_{kk} \int_t^{T_{k-1}} \delta_k^2 f^2(s, T_k) ds},\end{aligned}\quad (36)$$

where $c_{kk} = \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c(x, y) dx dy$. We know that correlation structure is independent of current time t , thus we have

$$\begin{aligned}\bar{\sigma}_{i,j}^{Black,RF} &= \sqrt{\frac{1}{T_i-t} \int_t^{T_i} \left\| \sum_{k=i+1}^j \frac{\delta_k L_k(s) \gamma_k^{i,j}(s)}{1 + \delta_k L_k(s)} \int_{T_{k-1}}^{T_k} f(s, T_k) dW(t, u) \right\|^2 ds} \\ &= \sqrt{\frac{1}{T_i-t} \sum_{k,l=i+1}^j \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} c_{kl} \delta_k \delta_l f(s, T_k) f(s, T_l) ds} \\ &= \sqrt{\frac{1}{T_i-t} \sum_{k,l=i+1}^j \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t) \gamma_l^{i,j}(t)}{1 + \delta_l L_l(t)} \sigma_k^{Black,RF} \sigma_l^{Black,RF} \frac{c_{kl}}{\sqrt{c_{kk} c_{ll}}} \Theta_{l,k}(t)},\end{aligned}\quad (37)$$

where $\Theta_{l,k}(t)$ is given in Eq.(35). The second equation above is obtained by using standard freezing approximation techniques, i.e. approximately evaluating the LIBOR rates $L_k(s)$, $t \leq s \leq T_i$, appearing in the instantaneous volatility at initial time t .

For RFLMM, from Eq.(13), we have that

$$\sigma_k^{Black,MR} = \sqrt{\frac{1}{T_{k-1}-t} \int_t^{T_{k-1}} \left[\int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} e(s, x) e(s, y) c_f(x, y) dx dy \right] ds}.$$

To facilitate the comparison between the Brownian motion and random field LMM, it is reasonable to set $e(t, x) = e(t, T_k)$ for $x \in [T_{k-1}, T_k]$. Thus the above equation becomes

$$\begin{aligned}\sigma_k^{Black,MR} &= \sqrt{\frac{1}{T_{k-1}-t} \int_t^{T_{k-1}} \delta_k^2 e^2(s, T_k) ds \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c_f(x, y) dx dy} \\ &= \sqrt{\frac{1}{T_{k-1}-t} c_{f,kk} \int_t^{T_{k-1}} \delta_k^2 e^2(s, T_k) ds},\end{aligned}\quad (38)$$

where $c_{f,kk} = \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c_f(x, y) dx dy$ and $c_{d,kk} = \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c_d(x, y) dx dy$.

Since the correlation structure is independent of current time t , from Eq.(14) we have

$$\begin{aligned}
& \overline{\sigma}_{i,j}^{Black,MR}(t) \\
= & \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^j \sum_{l=i+1}^j \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \left\{ \int_{T_{k-1}}^{T_k} \int_{T_{l-1}}^{T_l} [\alpha_k^{i,j}(s) \right.} \\
& \left. \xi_k(s, u) \alpha_l^{i,j}(s) \xi_l(s, v) c_d(s, u, v) + \beta_k^{i,j}(t) \eta_k(s, u) \beta_l^{i,j}(s) \eta_l(s, v) c_f(s, u, v) \right.} \\
& \left. + \alpha_k^{i,j}(s) \xi_k(s, u) \beta_l^{i,j}(s) \eta_l(s, v) c_{df}(s, u, v) \right.} \\
& \left. + \beta_k^{i,j}(s) \eta_k(s, u) \alpha_l^{i,j}(s) \xi_l(s, v) c_{df}(s, u, v) \right] dudv \} ds} \\
= & \sqrt{\frac{1}{T_i - t} \sum_{k=i+1}^j \sum_{l=i+1}^j \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \frac{\delta_l L_l(t)}{1 + \delta_l L_l(t)} \int_t^{T_i} \{ \delta_k \delta_l [\alpha_k^{i,j}(s) f(s, T_k) \right.} \\
& \left. \alpha_l^{i,j}(s) f(s, T_l) c_{d,kl} + \beta_k^{i,j}(t) e(s, T_k) \beta_l^{i,j}(s) e(s, T_l) c_{f,kl} + \alpha_k^{i,j}(s) f(s, T_k) \right.} \\
& \left. \beta_l^{i,j}(s) e(s, T_l) c_{df,kl} + \beta_k^{i,j}(s) e(s, T_k) \alpha_l^{i,j}(s) f(s, T_l) c_{df,kl} \right] dudv \} ds, \quad (39)
\end{aligned}$$

where $c_{i,kk} = \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c_i(x, y) dx dy$, for $i = d, f, df$.

We compute the root mean squared percentage pricing error for LIBOR caps as

$$RMSE(\%)_{caps} = \sqrt{\frac{1}{NN_0} \sum_{i=1}^{N_0} \sum_{j=1}^N \left(\frac{\sigma_j^{Black,RMF}(m_i) - \sigma_{i,j}^{Market}}{\sigma_j^{Black,MRF}(m_i)} \right)^2}, \quad (40)$$

where N is the number of tenor days and N_0 is the number of different cap strikes. Similarly, we compute the root mean squared percentage pricing errors for swaptions as

$$RMSE(\%)_{swaption} = \sqrt{\frac{2}{(N-1)(M-2)} \sum_{i=1}^M \sum_{j=i+1}^N \left(\frac{\overline{\sigma}_{i,j}^{Black} - \overline{\sigma}_{i,j}^{Market}}{\overline{\sigma}_{i,j}^{Black}} \right)^2}. \quad (41)$$

where N is the number of different tenors and M is the number of swaption maturities.

4.2 Data Description, Model Estimation and Pricing Performance

We first estimate the model parameters from the historical time series of the underlying interest rates. We then compute model prices of caps and

swaptions based on the estimated model parameters. Finally, we compare the difference between model price and market price.

Daily LIBOR forward rate curves, cap and swaption implied volatilities are obtained from Bloomberg from July 9, 2007 to October 9, 2009. We save about one year worth of data from October 15, 2008 to October 9, 2009 for out-of-sample pricing and use the preceding period from July 9, 2007 to October 14, 2008 for model estimation and in-sample pricing. Model parameters are estimated using unscented Kalman Filter which allows for non-linearity in state and transition equations.

Table 1 presents the estimation results of four different models, 1) single-curve LMM, 2) single-curve RFLMM, 3) two-curve LMM, and 4) two-curve RFLMM. Their details are described as follows.

1. Standard single-curve approach: we use the LIBOR standard yield curve to calculate the discount factors $P(t, T)$ and forward rates for projecting future swap cash flows. Uncertainties are driven by Brownian motions.
2. Random field single-curve approach: the discount curve and projecting swap cash flows is the same as in 1), i.e., the LIBOR standard yield curve, while uncertainties are modeled by Brownian fields.
3. Standard two-curve approach: we use OIS (overnight-index-swap) rates to calculate the discount factors $P(t, T)$ and use 6 month LIBOR forward curve to project swap cashflows, since the cap/floor and swaptions we are considering have tenor 6-month. The uncertainties are modeled as Brownian motions.
4. Random field two-curve approach: the same as 3), except that uncertainties are modeled by Brownian fields.

Table 1: Unscented Kalman Filter Estimation Results, July. 9, 2007–Oct. 14, 2008

Model	curve	a	b	c	d	ρ_∞	ρ_0
single-curve LMM		0.5543	1.2378	5.6365	1.275	2.7952	3.5513
		(0.022)	(0.0265)	(0.0763)	(0.0123)	(0.0124)	(0.1568)
single-curve RFLMM		0.5394	1.3457	5.4772	1.4685	8.9834	-
		(0.003)	(0.0446)	(0.167)	(0.0476)	(0.0252)	-
two-curve LMM	discount curve	0.5546	1.7655	5.8773	0.9754	2.6553	3.3572
		(0.032)	(0.0334)	(0.203)	(0.0104)	(0.0232)	(0.1292)
	projection curve	0.5523	1.2483	5.4772	1.4582	2.7834	3.527
		(0.032)	(0.0333)	(0.0693)	(0.0584)	(0.082)	(0.0632)
two-curve RFLMM	discount curve	0.5564	1.7834	5.9823	0.8468	8.3834	-
		(0.022)	(0.0634)	(0.332)	(0.0592)	(0.0134)	-
	projection curve	0.5342	1.3452	5.3343	1.4646	8.9652	-
		(0.016)	(0.0393)	(0.224)	(0.0633)	(0.0124)	-

Table 2 presents the in-sample pricing results of caps of various maturities by the four models. Across all maturities, we observe that during this sample period of the credit crisis, random field models produce smaller

Table 2: ATM Cap Valuation %RMSEs(July 9, 2007-October 14, 2008)

	1	2	3	4	5	6	7	8	9
single-curve	0.5047	0.4846	0.4746	0.4665	0.4575	0.4479	0.4176	0.3945	0.3532
LMM	(0.2516)	(0.1749)	(0.1427)	(0.1214)	(0.1014)	(0.0927)	(0.0893)	(0.0893)	(0.0885)
two-curve	0.4866	0.4656	0.4548	0.4454	0.4225	0.4012	0.3760	0.3553	0.3312
LMM	(0.1596)	(0.1658)	(0.1383)	(0.1192)	(0.1085)	(0.0942)	(0.0910)	(0.0914)	(0.0899)
single-curve	0.3458	0.3137	0.2909	0.2710	0.2521	0.2346	0.2005	0.1708	0.1337
RFLMM	(0.2512)	(0.1412)	(0.1280)	(0.1167)	(0.1028)	(0.0979)	(0.0962)	(0.0992)	(0.0998)
two-curve	0.3256	0.2931	0.2698	0.2490	0.2166	0.1858	0.1557	0.1261	0.1184
RFLMM	(0.1153)	(0.1264)	(0.1200)	(0.1118)	(0.1065)	(0.0975)	(0.0965)	(0.0999)	(0.1001)

%RMSEs than the corresponding Brownian motion models. Moreover, two-curve models out-perform the corresponding single curve models in producing smaller %RMSEs. The reason could be that during the credit crisis the rates used for discounting and projecting future swap cash flows have non-negligible spreads.

Similar observations can also be made for the time series of both in-sample pricing (Figure 3) and out-of-sample pricing (Figure 4) of ATM swaptions.

Table 3: ATM Swaption Valuation %RMSEs for two-curve RFLMM(July 9, 2007-October 14, 2008)

maturities (years)	lengths(year)								
	1	2	3	4	5	6	7	8	9
0.5	1.0785 (0.1129)	1.0707 (0.0572)	1.0695 (0.0358)	1.0711 (0.0255)	1.0751 (0.0194)	1.0762 (0.0161)	1.0790 (0.0139)	1.0809 (0.0122)	1.0829 (0.0109)
1	1.0228 (0.1123)	1.0220 (0.0615)	1.0214 (0.0415)	1.0247 (0.0311)	1.0289 (0.0252)	1.0317 (0.0210)	1.0348 (0.0186)	1.0375 (0.0165)	1.0405 (0.0149)
2	0.9368 (0.0993)	0.9409 (0.0628)	0.9455 (0.0463)	0.9518 (0.0380)	0.9581 (0.0315)	0.9613 (0.0279)	0.9651 (0.0247)	0.9692 (0.0223)	
3	0.8727 (0.0906)	0.8817 (0.06110)	0.8886 (0.0509)	0.8932 (0.0415)	0.8999 (0.0359)	0.9050 (0.0314)	0.9105 (0.0280)		
4	0.8202 (0.0880)	0.8287 (0.07660)	0.8361 (0.05750)	0.8443 (0.0473)	0.8480 (0.0402)	0.8547 (0.0353)			
5	0.7716 (0.2296)	0.7820 (0.0958)	0.7921 (0.0656)	0.7961 (0.0524)	0.8041 (0.0444)				
7	0.6903 (0.1063)	0.6972 (0.0752)	0.7052 (0.0603)						

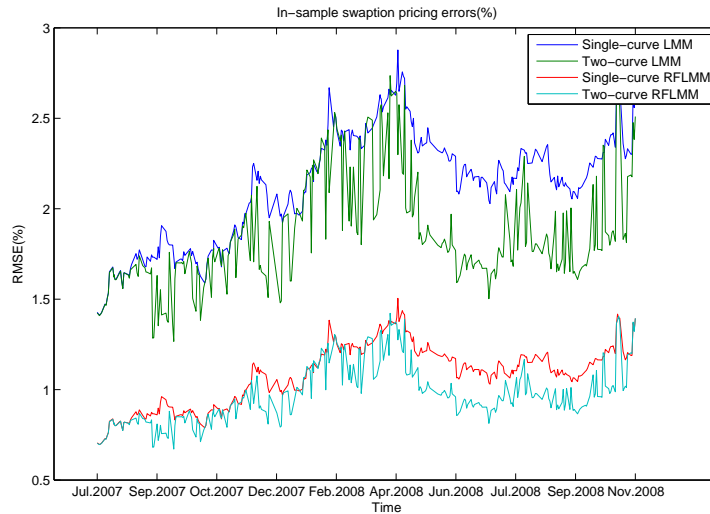


Figure 3: Time series of %RMSE of Swaptions for single-curve LMM, RFLMM and two-curve LMM, RFLMM over the period Jul.07-Oct.08(In-sample pricing)

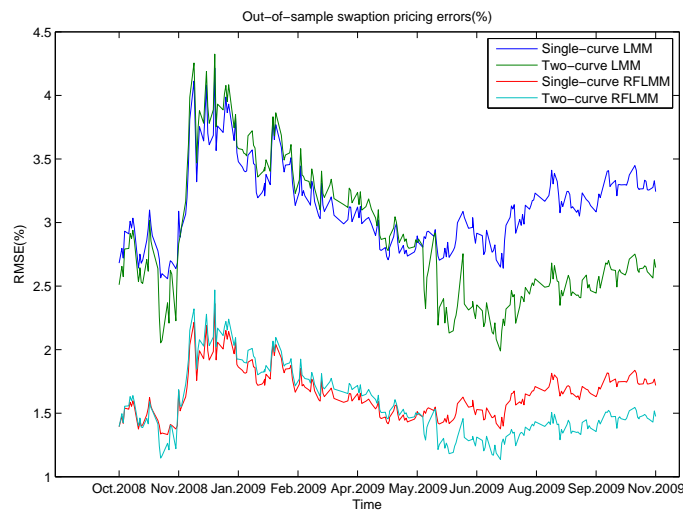


Figure 4: Time series of %RMSE of Swaptions for single-curve LMM, RFLMM and two-curve LMM, RFLMM over the period Oct.08-Oct.09(Out-of-sample pricing)

5 Conclusions

This paper extends the LIBOR market model to the multi-curve framework. In addition, uncertainties are driven by random field. In Sec.3.1, we extend the LIBOR market model with multi-curve framework to the random field case (RFLMM), where the innovation terms are modelled as Gaussian random field. Closed-form solutions for caplet and swaption prices are given in Sec.3.2. In Sec.3.3, we derive the local volatility smile models with multi-curve framework in random field case. The approximation formulas of implied volatilities are obtained. Sec.3.4 discusses stochastic volatility models, in particular, the SABR model and Wu-Zhang model.

From the empirical results we observe that random field models are more accurate than Brownian motion models in pricing. In addition, the multi-curve (two-curve) models produce smaller pricing errors than single-curve models for both models driven by Brownian motions and models driven by random fields.

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