Hedging forward positions: Basis risk versus liquidity costs

Stefan Ankirchner, Peter Kratz, Thomas Kruse

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Companies operating a gas power plant have an immanent
- short forward position of natural gas (NG),
- long forward position of power.

To reduce price risk they
- buy natural gas on forward markets,
- sell power on forward markets.

Suppose that a German energy company wants to buy today the NG it needs for the second half of 2014.

Problem: German gas forward market is very illiquid.
• Bid-ask-spread ↓ as time to delivery approaches
• Dutch and German gas prices are **highly** correlated
2 Ways of Hedging

▶ Hedge 1:
Buy natural gas in G

▶ Hedge 2:
Buy natural gas in NL.
Shortly before delivery: sell in NL and buy in G.

Pros & Cons:

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<td>No risk</td>
<td>Low liqu. costs</td>
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<td>Con</td>
<td>High liqu. costs</td>
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Trade-off: High liquidity costs versus basis risk

Question: What is the optimal position in German and Dutch NG at any time before July 2014?

⇒ A singular (stochastic) control problem
initial short position : $x_0 < 0$

$T =$ time horizon

$X_t =$ primary asset position (e.g. German NG);

Constraints: $X_{0^-} = x_0$ and $X_T = 0$

$Y_t =$ proxy position (e.g. Dutch NG)

Constraints: $Y_{0^-} = 0$ and $Y_T = 0$
Minimizing overall costs ⇔ minimizing execution costs

- $P_t =$ forward price of the primary asset at time $t$ (a continuous martingale)
- $K_t =$ liquidity costs of primary asset at time $t$ (a non-negative process with cadlag paths)
- $L =$ half bid-ask-spread of proxy

**Expected costs in the primary asset:**

$$C^1(X) = E \left[ \int_{[0, T]} P_s dX_s + \int_{[0, T]} K_s |dX_s| \right] = -P_0 x_0 + E \left[ \int_{[0, T]} K_s |dX_s| \right].$$

**Expected costs in the proxy:**

$$C^2(Y) = E \left[ \int_{[0, T]} L |dY_s| \right]$$

**Expected execution costs**

$$C(X, Y) = E \left[ \int_{[0, T]} K_s |dX_s| + \int_{[0, T]} L |dY_s| \right]$$
The model cont’d

Risk

- $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$ covariance matrix
- Instantaneous risk at time $t$:
  $$f(X_t, Y_t) = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}^T \Sigma \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$$
- Overall risk:
  $$R(X, Y) = \int_0^T \sqrt{f(X_t, Y_t)} \, dt$$
Control problem:

\[ C(X, Y) + \lambda R(X, Y) \longrightarrow \min! \]

Lemma

Let \( X \) be a given primary position path and assume that \( L = 0 \). Then the optimal cross hedge is given by

\[ Y_t^* = -\rho \frac{\sigma_1}{\sigma_2} X_t. \]

\( h = \rho \frac{\sigma_1}{\sigma_2} = \text{minimum variance hedge ratio} \)
Proposition

Let $(X^*, Y^*)$ be optimal. Then almost surely

a) $X_t^*$ is non-decreasing, and

b) there exists a càdlàg, adapted and non-decreasing process $I$ such that $Y_t^* = I_t \wedge -hX_t^*$. 

![Graph showing optimal position paths](image-url)
Our method for getting explicit solutions

**Assumption:** The optimal cross hedge $Y(X)$ associated to any $X$ is non-increasing after 0, i.e. of the form

$$Y(X)_t = y \wedge -\rho \frac{\sigma_1}{\sigma_2} X_t.$$ 

**Iterative Method:**

1. For a given $y \geq 0$ determine the optimal primary position $X = X(y)$. To this end reformulate the problem as a stopping problem.
2. Determine optimal initial cross hedge position $y^*$. 
3. The optimal positions are given by

$$X_t^* = X_t(y^*) \text{ and } Y_t^* = y^* \wedge -\rho \frac{\sigma_1}{\sigma_2} X_t^*.$$
The primary position via optimal stopping

For any \( y \geq 0 \) consider the problem

\[
E \left[ \int_{[0,T]} K_s dX_s + \int_0^T g(X_s) ds \right] \rightarrow \text{min!} \tag{1}
\]

where \( g(x) = \lambda \sqrt{f(x, y \wedge -\rho \frac{\sigma_1}{\sigma_2} x)} \).

Proposition

For all \( x \in [x_0, 0] \) let \( \tau(x) \) be the solution of the stopping problem

\[
\inf_{\tau \in \mathcal{T}_{0,T}} E\left[ K_\tau + \tau g'(x) \right]
\]

\text{marginal cost} + \text{marginal risk}

Then an optimal primary position \( X \) for (1) is given by

\[ X_t = \inf \{ x \in [x_0, 0] | \tau(x) > t \} \].
The primary position via optimal stopping

Proof

Right continuous inverse of a position path:

\[ \tau(x) = \inf \{ t \geq 0 | X_t > x \} \]
The primary position via optimal stopping

Proof

Right continuous inverse of a position path:

$$\tau(x) = \inf \{ t \geq 0 | X_t > x \}$$

Apply a Change of Variables Formula to the cost term

$$\int_{[0,T]} K_s \, dX_s = \int_{x_0}^{0} K_{\tau(z)} \, dz$$
The primary position via optimal stopping

Proof

The risk term satisfies

\[ \int_0^T g(X_s) ds = \int_0^T \int_{X_0}^{X_s} g'(z) dz + g(x_0) ds \]

\[ = \int_0^{X_0} \tau(z) g'(z) dz + g(x_0) T \]

Hence

\[ E \left[ \int_{[0,T]} K_s dX_s + \int_0^T g(X_s) ds \right] = \int_{x_0}^0 E \left[ K_{\tau(z)} + \tau(z) g'(z) \right] dz + g(x_0) T \]

\textit{marginal cost + marginal risk}

\[ \rightarrow \text{Minimize } \textit{marginal costs} + \textit{marginal risk} \text{ pointwise} \]
Example: Convex deterministic costs

- Liquidity costs are *deterministic*, decreasing and convex in time
- $L = 0 \rightarrow$ marginal risk is constant in $x$
- marginal risk increases linearly in time
- $t \mapsto K_t + tg'(x)$ is convex
Example: Convex deterministic costs

- Liquidity costs are *deterministic*, decreasing and convex in time.
- $L = 0 \rightarrow$ marginal risk is constant in $x$
- marginal risk increases linearly in time
- $t \mapsto K_t + tg'(x)$ is convex

$\exists$ optimal turning point $t^*$
Proposition

Suppose that $L = 0$ and that $K \in C^1$ is decreasing and convex on $[0, T]$. If $\lambda \sigma_1 \sqrt{1 - \rho^2} \in [-\dot{K}(T), -\dot{K}(0)]$, then the optimal closing time is given by

$$t^* = (\dot{K})^{-1}(-\lambda \sigma_1 \sqrt{1 - \rho^2}),$$

and $X^* = x_0 1_{[0,t^*)}$ and $Y^* = -\rho \frac{\sigma_1}{\sigma_2} x_0 1_{[0,t^*)}$ are the optimal position processes.
Example: Concave deterministic costs

- $L \geq 0$
- Liquidity costs in primary asset are \textit{deterministic} and concave
- $t \mapsto K_t + tg'(x)$ is concave

$\rightarrow$ Optimal strategies are static
Example cont’d: Optimal strategies are static

Proposition

Suppose that $K$ is decreasing and concave on $[0, T]$. Then the optimal position strategy is of the form

$$X_t^* = x^* 1_{[0, T]}(t) \quad \text{and} \quad Y_t^* = y^* 1_{[0, T]}(t),$$

with $x^* \leq 0$ and $y^* \geq 0$.

The optimal positions $x^*$ and $y^*$ can be calculated explicitly (tedious!).
Example: Active trading kicks in at a random time

- $K$ jumps at a random time $\tau$ from a higher level $K_+$ to a lower level $K_-$.  
- $\tau$ is the first jump time of an inhomogeneous Poisson process with non-decreasing jump intensity.

$\rightarrow$ Close positions at time $\tau$: $X_s = Y_s = 0$ for all $s \geq \tau$.  

\[
\text{liquidation costs}
\]

\[
\text{time}
\]
Example cont’d: Optimal strategies are static

Proposition
Suppose that $K$ jumps from $K_+$ to $K_-$ at time $\tau$. Then the optimal position strategy is of the form

$$X_t^* = x^* 1_{[0,\tau)}(t) \quad \text{and} \quad Y_t^* = y^* 1_{[0,\tau)}(t),$$

with $x^* \leq 0$ and $y^* \geq 0$.

The optimal positions $x^*$ and $y^*$ can be calculated explicitly.
Example cont’d: Decision tree

\[ L \geq \bar{L} \]

- yes
  - keep primary open
  - cross hedge with \( A \) forwards

- no
  - close primary
  - do not cross hedge

\[ \Delta K \geq \lambda \sigma_1 E[\tau] \]

- yes
  - keep primary open
  - do not cross hedge

- no
  - close primary
  - do not cross hedge

\[ \bar{L} = \frac{\sigma_2}{2\sigma_1} \left( \Delta K \rho - \sqrt{(1 - \rho^2)(\lambda^2 \sigma_1^2 E[\tau]^2 - \Delta K^2)^+} \right) \]

\[ A = -\frac{\sigma_1}{\sigma_2} \max \left( 0, \rho - 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{(\lambda^2 \sigma_2^2 E[\tau]^2 - 4L^2)^+}} x_0 \right) \]
Conclusion

- When hedging on forward markets one frequently has to choose between *liquidity costs* and *basis risk*.
- We introduce a **singular control model** allowing to characterize optimal trade-offs.
- Optimal position paths can be obtained by solving *families of stopping problems*.
- For specific examples we present optimal position paths in closed form.
Thank you!