Illiquidity Contagion and Liquidity Crashes*

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Abstract

Liquidity providers in a security often learn information from the prices of other securities. We show that cross-asset learning generates a self-reinforcing positive relationship between price informativeness and liquidity. This relationship causes liquidity spillovers and is a source of fragility: a small drop in the liquidity of one security can, through a feedback loop, result in a very large drop in market liquidity and price informativeness (a liquidity crash). It also generates multiple equilibria characterized either by high illiquidity and low price informativeness or low illiquidity and high price informativeness. The model suggests a new explanation for co-movements in liquidity and liquidity dry-ups.

Keywords: Liquidity spillovers, Contagion, Liquidity Crashes, Multiple equilibria, Rational expectations

JEL Classification Numbers: G10, G12, G14

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1 Introduction

Fluctuations in market liquidity play a crucial role in financial markets. In particular, commonal-
ities in liquidity (co-movements in liquidity among assets) and liquidity dry-ups matter for asset pricing. Yet, the causes of these variations are not well understood. One possibility is that common shocks (e.g. correlated demands shocks or market-wide tightening of funding constraints for liquidity suppliers) affect the demand or the supply of liquidity across assets. Another possibility, which has received less attention, is that shocks specific to liquidity supply in one asset class propagate to other asset classes.

How can this happen? What is the mechanism for such liquidity spillovers? In this paper we propose an answer to this question. Liquidity providers in one asset class (e.g., dealers in ETFs) often learn information from the prices of other assets (e.g., assets underlying ETFs). We show that cross-asset learning makes liquidity of different assets interconnected: if the liquidity of one asset drops, its price becomes less informative and as a result the liquidity of other assets drops as well.

To see this intuitively, consider dealers in one asset, $X$, who use the price of another asset, $Y$, as a source of information. Fluctuations in the price of $Y$ are informative because they reflect information about fundamentals known to dealers in this asset. However, this signal is noisy since price fluctuations also reflect transient price pressures due to uninformed trades. These transient price pressures account for a larger fraction of price volatility when the cost of liquidity provision is higher. For this reason, the informativeness of the price of asset $Y$ for dealers in asset $X$ is smaller when asset $Y$ is less liquid.

Now suppose that a shock specific to asset $Y$ increases the cost of liquidity provision for dealers in this security (e.g., their risk appetite declines). Thus, asset $Y$ becomes less liquid and, for this reason, its price becomes less informative. As a result, uncertainty for dealers in asset $X$ is higher and the cost of liquidity provision for these dealers increases as well. In this way, the drop in liquidity for asset $Y$ propagates to asset $X$, as shown in Figure. In turn, this spillover makes the price of asset $X$ less informative for dealers in asset $Y$, sparking a chain reaction that amplifies

\footnotetext[1]{Evidence of co-variations in liquidity and the sources of these covariations are provided in Chordia et al. (2000), Hasbrouck and Seppi (2001), Huberman and Halka (2001), Coughenour and Saad (2004), Chordia et al. (2005), Kamara, Lou, and Sadka (2008), Korajczyk and Sadka (2008), Comerton-Forde et al. (2010), or Karolyi, Lee, and van Dijk (2011). For implications for asset pricing, see Pastor and Stambaugh (2003), Acharya and Pedersen (2005), Korajczyk and Sadka (2008) and Amihud et al. (2005) for a survey.

\footnotetext[2]{See Karolyi, Lee, and van Dijk (2012) for a survey and an empirical evaluation using international data of the various explanations for liquidity co-movements.

\footnotetext[3]{For stocks listed on the NYSE, Hendershott, Li, Menkveld and Seasholes (2010) show that 25% of the monthly return variance is due to transitory price changes. Interestingly, they also find that transient price pressures are stronger when market-makers’ inventories are relatively large. This finding implies that price movements are less informative when dealers’ cost of liquidity provision is higher, in line with our model.

\footnotetext[4]{We measure illiquidity by the sensitivity of asset prices to market order imbalances, as in Kyle (1985) for instance. Illiquidity is higher when this sensitivity is high. This will be the case if the limit order book of a security is thin. Empirically, this sensitivity is often measured by regressing price changes on order imbalances (see for instance Glosten and Harris (1988) or Korajczyk and Sadka (2008)).}
the initial shock.

Figure 1: Cross-asset learning and liquidity spillovers.

We formalize this illiquidity spillover mechanism in a two-assets rational expectations model where assets have a two-factor structure. In the baseline version of our model, there is no cross-asset trading so that, by construction, effects can only arise from cross-asset learning (rather than hedging effects). Dealers specialized in one asset are well-informed on one risk factor and learn information on the other risk factor from the price of the other security. This information structure yields the spillover mechanism and the feedback loop portrayed in Figure 1.

This feedback loop generates an illiquidity multiplier: a small shock to the illiquidity of one asset (e.g., a small increase in the risk aversion of dealers in this asset) is amplified, so that, its final effect on the illiquidity of each asset is larger than its immediate effect. There always exist parameter values such that this illiquidity multiplier is very high.

These findings have important implications. First, changes in market structure (e.g., a change of trading system) or regulations directly affecting the liquidity of one asset class should also have significant effects on the liquidity of other assets. Second, when the illiquidity multiplier is large, liquidity is fragile: a small increase in illiquidity for one asset can eventually trigger a very large increase in illiquidity for all assets. Thus, the model can generate liquidity crashes, that is, market wide evaporations of liquidity, in the absence of noticeable changes in the economic environment or asset prices.

5This prediction is consistent with the empirical findings of Amihud and Mendelson (1997) and Bessembinder et al. (2006).

The 2010 Flash crash shows that a liquidity crash is not only a theoretical possibility. In a few minutes after 2:30 p.m. on May 6, 2010, limit order books for various asset classes (index futures, ETFs, and hundreds of individual stocks) became very thin (see Figure 1 in CFTC-SEC (2011)), both on the buy and sell sides. Furthermore, the liquidity dry-up for ETFs and their underlying securities preceded the decline in prices for these securities (see Borkovec et al. (2010)). This suggests that the drop in liquidity during the Flash crash is due, at least partly, to liquidity suppliers’ decision to curtail their liquidity provision (e.g., by cancelling limit orders), rather than a mechanical consumption of liquidity due to a market-wide selling pressure (i.e., correlated demand shocks).

In our model, the illiquidity multiplier is high when the sensitivity of an asset liquidity to the liquidity of the other asset (a measure of the “strength” of liquidity spillovers) is high. This suggests to use these sensitivities for a set of assets to build early indicators of their exposure to a liquidity crash. For instance, future empirical work could check whether liquidity spillovers in U.S. markets were particularly strong in the hours or days preceding the Flash crash.

The key mechanism behind our amplification mechanism is that liquidity and price informativeness reinforce each other: liquidity of all assets is higher when prices are more informative and vice versa. Thus, the model predicts that liquidity crashes should be associated with sharp drop in price informativeness. This complementarity also gives rise to multiple equilibria with differing levels of liquidity and price informativeness. For instance, if dealers expect the prices of other assets to contain little information, they primarily rely on their own information and face more uncertainty. Liquidity is then low in all assets and prices are very noisy, which validates dealers’ expectations. Thus, dealers’ beliefs about the informativeness of prices and liquidity for other securities are self-fulfilling. Consequently, there exist cases in which, for the same parameter values, liquidity and price informativeness are either relatively high for all assets or relatively low for all assets.

Multiplicity of equilibria is another form of fragility. Indeed, illiquidity can experience a significant jump for all assets, without any apparent reason, when there is a switch from the equilibrium with high liquidity to the equilibrium with low liquidity. Our explanation for liquidity crashes however does not rely on this mechanism: in a given equilibrium, the illiquidity multiplier can be very high whether or not the equilibrium is unique.

In order to obtain additional implications, we consider two extensions of our baseline model. First, we study the effect of varying the fraction of dealers in one asset with access to price information about the other asset. We refer to these dealers as “pricewatchers.” This extension...
generates two testable predictions. First, co-movements in liquidity should be stronger when the fraction of pricewatchers is larger. This prediction could be tested by considering technological shocks enabling dealers to condition their strategies on a wider array of prices.\footnote{Easley, Hendershott, and Ramadorai (2009) analyze the effect on risk premia of a technological change improving the dissemination of quote information on the floor of the NYSE in 1980. Our model predicts that this change should have strengthened the covariation in liquidity of NYSE stocks.} Second, we show that an increase in the fraction of pricewatchers is itself a source of liquidity spillovers. When more dealers in one asset class become better informed about other asset prices then the liquidity of the former asset class increases, which in turn improves the liquidity of other assets.

Second, we consider the possibility of cross-asset trading by arbitrageurs. Arbitrageurs often respond to a demand for liquidity in one asset by hedging their position in other assets to reduce the risk of their portfolios (see Gromb and Vayanos (2010) for a survey). Hence, they are often viewed as one channel through which shocks to demand for one asset propagate to other assets.\footnote{See Greenwood (2005) and Ben David, Franzoni, and Moussawi (2011) for empirical analyses.} This effect is present in our model. However, we show that cross-market arbitrage dampens the illiquidity multiplier rather than increasing it. As a result, the size of co-movements in liquidity between two assets decreases with the capital allocated to cross-market arbitrage in these assets, which is another testable implication of the model.

Our analysis is related to models of financial crises (e.g., Gennotte and Leland (1990), Barlevy and Veronesi (2003), and Yuan (2005)) and financial contagion (e.g., King and Wadhwani (1990), Kodres and Pritsker (2002), or Pasquariello (2007)).\footnote{Models of contagion often build upon the multi-asset pricing models of Admati (1985) or Caballe and Krishnan (1994).} Some channels for the propagation of price declines highlighted by these theories are also present in our model. Specifically, a drop in the price of one asset triggers a drop in the price of the other asset because (i) this drop conveys negative information and (ii) cross-market trades (portfolio rebalancing) by cross-market arbitrageurs transmit price pressures exerted in one market to another.\footnote{The literature has identified other channels for cross-market price impacts (see Kodres and Pritsker (2002) for a discussion). Several papers empirically study mechanisms through which price pressures in one security propagates to other securities. See, for instance, Greenwood (2005), Andrade, Chang and Seasholes (2008), Boulatov, Hendershott and Livdan (2010), or Pasquariello and Vega (2012).} Formalizing these well-known channels is not the contribution of our paper. Rather, our contribution is to show how a drop in the liquidity of one asset class can propagate to another asset class and how liquidity can evaporate for multiple assets at the same time, using the standard CARA-normal framework. This is of interest since this framework has been widely used to study price informativeness and financial contagion but its implications for liquidity fluctuations have not attracted much attention.

Last, to our knowledge, the possibility of multiple equilibria due to a self-reinforcing loop between price informativeness and liquidity for different assets is a novel finding. In Gennotte and Leland (1990), Barlevy and Veronesi (2003), and Yuan (2005), there might be multiple clearing prices for given realizations of investors’ signals and the asset supply because investors’ aggregate...
The demand curve is backward bending over some price ranges.\footnote{This property of the aggregate demand curve arises for different reasons in each of these models.} In contrast, in our model, aggregate demand curves are everywhere linearly decreasing in prices in equilibrium. Goldstein and Yang (2012) considers a single asset model in which two groups of investors receive different pieces of information about the asset. When the weights of each group in the economy are close enough, they show that the trading intensity of both groups of investors reinforces each other, due to an “uncertainty reduction effect.” This effect is similar to the mechanism that leads to positive liquidity spillovers across assets in our model but, in their model, the equilibrium is unique.

As already mentioned, fluctuations in financing constraints (e.g., Gromb and Vayanos (2002), or Brunnemeier and Pedersen (2008)) or wealth effects (e.g., Kyle and Xiong (2001)) are possible sources of covariation in liquidity supply. These mechanisms however require either a common shock to funding constraints in multiple assets or common liquidity suppliers across assets. In contrast, co-variations in liquidity supply in our model arise from idiosyncratic shocks to one asset liquidity. This difference offers one way to distinguish empirically these two explanations for co-movements in liquidity supply.

The rest of the paper is organized as follows. Section 2 describes the model. In Section 3, we consider the baseline case in which liquidity providers are specialized in one asset and have all access to price information. We show how cross-asset learning generates illiquidity spillovers and is a source of multiple equilibria. In Section 4 we consider two extensions: (i) the case in which only a fraction of dealers have access to price information and (ii) the case in which some liquidity providers (“cross-market arbitrageurs”) can operate in both securities. Section 5 concludes. Proofs are collected in the Appendix.

2 The model

We consider a two periods model with two assets, \(D\) and \(F\). The payoffs of the assets at date 2, \(v_D\) and \(v_F\), are

\[
\begin{align*}
v_D & = \delta_D + d_D \times \delta_F + \eta_D, \\
v_F & = d_F \times \delta_D + \delta_F + \eta_F.
\end{align*}
\]

(1)

(2)

The risk factors, \(\delta_j\), and idiosyncratic risks, \(\eta_j\), of the two assets are normally and independently distributed with mean zero and variances, \(\sigma^2_{\delta_j}\) and \(\sigma^2_{\eta_j}\). We normalize the variance of each factor to one (\(\sigma^2_{\delta_j} = 1\) for \(j \in \{D, F\}\)).

Trade takes place at date 1 between liquidity traders and dealers. Liquidity traders’ aggregate demand for asset \(j\) is \(u_j \sim N(0, \sigma^2_{u_j})\), with \(E(u_D u_F) = 0\) and \(E(u_j u_k) = 0\) for \(j, k \in \{D, F\}\). Dealers provide liquidity by taking the opposite side of liquidity traders’ transaction and are specialized: they trade either asset \(D\) or asset \(F\). Schultz (2003) shows empirically that dealers specialize in
stocks in which they have an informational advantage. Thus, we assume that dealers specialized in security \( j \) have perfect information on factor \( \delta_j \) and no information on factor \( \delta_{-j} \) (if \( j = D \) then \( -j = F \) and vice versa).\(^{14}\)

We denote by \( p_j \) the price of asset \( j \) at date 1. Prices will reflect dealers’ information (see below). Hence, a dealer specialized in one asset can learn information from the price of the other asset (“cross-asset learning”). We denote by \( \mu_j \) the fraction of dealers specialized in security \( j \) who observes the price of security \(-j\). We refer to these dealers as being \textit{pricewatchers}. We use \( W \) to index the decisions made by pricewatchers and \( I \) to index the decisions made by other dealers (index “\( I \)” stands for “Inattentive”).\(^{15}\)

Each dealer in asset \( j \) has a CARA utility function with risk tolerance \( \gamma_j \). Thus, if dealer \( i \) in asset \( j \) holds \( x_{ij} \) shares of this asset, her expected utility is

\[
E \left[ U \left( \pi_{ij} \mid \delta_j, \mathcal{P}_j^k \right) \right] = E \left[ -\exp \left\{ -\gamma_j^{-1} \pi_{ij} \right\} \right],
\]

where \( \pi_{ij} = (v_j - p_j)x_{ij} \) and \( \mathcal{P}_j^k \) is the \textit{price information} available to a dealer with type \( k \in \{W, I\} \) operating in security \( j \), that is, \( \mathcal{P}_j^W = \{ p_j, p_{-j} \} \) and \( \mathcal{P}_j^I = \{ p_j \} \). Pricewatchers’ optimal position in asset \( j \) is denoted by \( x_j^W(\delta_j, p_j, p_{-j}) \). The optimal position of inattentive dealers in asset \( j \) is denoted by \( x_j^I(\delta_j, p_j) \) (we drop subscript \( i \) since equilibrium strategies are symmetric).

As each asset is in zero net supply, the market clearing condition for asset \( j \) is

\[
\mu_j x_j^W(\delta_j, p_j, p_{-j}) + (1 - \mu_j)x_j^I(\delta_j, p_j) + u_j = 0, \quad \text{for } j \in \{D, F\}.
\]

As is common (see Kyle (1985)), we measure illiquidity of security \( j \) by the sensitivity of its price to liquidity demand, that is, \( \partial p_j / \partial u_j \). Security \( j \) is less illiquid if its price decreases (increases) less when \( u_j < 0 \) (\( u_j > 0 \)).

We assume that dealers are specialized in one asset in order to better highlight the role of cross-asset learning (inferences that dealers make from the prices of other assets) in propagating and amplifying asset specific illiquidity shocks (shocks on \( \partial p_j / \partial u_j \)). Our assumption turns off cross-market hedging effects (dealers’ diversification of risk by trading multiple assets), leaving cross-asset learning as the only possible channel for the propagation of illiquidity shocks. This assumption is also consistent with the fact that some market-making desks are specialized.\(^{16}\)

\(^{14}\)Kumar and Seppi (1994) consider a model of arbitrage between a index futures and the stocks in the index with similar assumptions for dealers (they only trade in one market and have specialized information). In contrast to Kumar and Seppi (1994), at least some dealers in one asset observe the price of the other asset without delay in our model.

\(^{15}\)In reality, some dealers may choose to ignore the prices of some assets even though these prices contain information because observing these prices is difficult or costly. For instance, the SEC-CFTC report on the Flash crash notes that “some ETF market-makers and liquidity providers treat ETFs as if they were the same as corporate stocks and do not track the prices of the individual securities underlying the ETFs... Others heavily depend upon the tracking of underlying securities as part of their ETF pricing... and yet others trade in individual securities at the same time they trade ETFs.” (see page 39, SEC-CFTC (2011)). The first group corresponds to our inattentive dealers, the second to pricewatchers, and the third group to the cross-market arbitrageurs analyzed in Section 4.2.

\(^{16}\)In reality, dealer firms are active in multiple securities. However, these firms delegate trade-related decisions to...
we consider the more general case in which specialized market-makers coexist with “cross-market arbitrageurs,” who provide liquidity in both assets and engage in cross-market hedging. All the findings of the baseline case are robust to this extension.

3 Cross-asset learning and liquidity spillovers

In this section, we derive the main findings of the paper when \( \mu_D = \mu_F = 1 \). We then consider the effects of varying the fraction of pricewatchers in Section 4.

3.1 Benchmark: Fully segmented markets

As a benchmark, it is useful to consider the case in which \( \mu_D = \mu_F = 0 \). In this case, the markets for the two assets are fully segmented since (a) all dealers are specialized and (b) they do not pay attention to the price of the other asset.

Lemma 1. (Benchmark) When \( \mu_F = \mu_D = 0 \), the equilibrium price in market \( j \) is:

\[
p_j = \delta_j + B_{j0} u_j,
\]

(5)

where \( B_{j0} = \gamma_j^{-1} \text{Var}[v_j|\delta_j] = \gamma_j^{-1}(\sigma^2_{\eta_j} + d_j^2) \) is the illiquidity of asset \( j \in \{D, F\} \).

The illiquidity of a security decreases with the risk appetite of the dealers specialized in this security (\( \gamma_j \)) and increases with the uncertainty on the payoff of this security (\( \text{Var}[v_j|\delta_j] = (\sigma^2_{\eta_j} + d_j^2) \)). Parameters \( \gamma_j, \sigma^2_{\eta_j} \) and \( d_j \) determine the illiquidity of security \( j \) and have no effect on the illiquidity of the other security. Thus, we refer to these parameters as being the “liquidity fundamentals” of security \( j \).

In the benchmark case, the equilibrium is unique and there are no liquidity spillovers: a change in the liquidity fundamental of one market does not affect the illiquidity of the other market. For instance, an increase in the risk tolerance of dealers in security \( D \) makes this security more liquid without affecting the illiquidity of the other security.\(^{17}\)

3.2 Interconnected liquidity

We now consider the polar case in which all dealers are pricewatchers (\( \mu_D = \mu_F = 1 \)). The analysis is more complex than in the benchmark case as dealers in one security extract information about individuals who operate on specialized trading desks. Naik and Yadav (2003) show empirically that the decision-making of these trading desks is largely decentralized (e.g., dealers’ trading decisions within a firm are mainly driven by their own inventory exposure rather than the aggregate inventory exposure of the dealer firm to which they belong). Their results suggest that there is no direct centralized information sharing between dealers within these firms. See also CFTC-SEC (2011), pp. 37–38.

\(^{17}\)A shock to the risk tolerance of dealers in one security is just one way to exogenously vary the cost of liquidity provision for dealers in this asset. In reality variations in this cost may also be due to variations in risk limits, or financing constraints for dealers. The important point is that they do not directly affect dealers in other assets.
the risk-factor unknown to them from the price of the other security. To solve this signal extraction problem, dealers must form beliefs on the relationship between clearing prices and risk factors. We focus on equilibria in which these beliefs are correct, that is, the rational expectations equilibria of the model.

A linear rational expectations equilibrium is a set of prices \( p_{j1}^* \), \( j \in \{D,F\} \) such that

\[
p_{j1}^* = R_{j1} \delta_j + B_{j1} u_j + E_{j1} \delta_{j-} + C_{j1} u_{-j},
\]

and \( p_{j1}^* \) clears the market of asset \( j \) for each realization of \( \{u_j, \delta_j, u_{-j}, \delta_{-j}\} \) when dealers anticipate that clearing prices satisfy equation (6) and choose their trading strategy to maximize their expected utility (given in equation (3)). We say that the equilibrium is non-fully revealing if dealers in security \( j \) cannot perfectly infer the realization of risk factor \( \delta_{-j} \) from the price of security \( -j \). The illiquidity of market \( j \) is measured by \( B_{j1} = \partial p_j/\partial u_j \). Index “1” refers to the equilibrium when \( \mu_D = \mu_F = 1 \).

**Proposition 1.** When \( \mu_D = \mu_F = 1 \) and \( \sigma_{\eta_D} > 0 \) or \( \sigma_{\eta_F} > 0 \), there always exists either one or three non-fully revealing linear rational expectations equilibria. At equilibrium the levels of illiquidity of assets \( D \) and \( F \), \( B_{D1} > 0 \) and \( B_{F1} > 0 \), solve:

\[
B_{D1} = f_{D1}(B_{F1}; \gamma_D, \sigma_{\eta_D}, d_D, \sigma_{u_F}) = \frac{\sigma_{\eta_D}^2}{\gamma_D} + \frac{d_D^2 B_{F1}^2 \sigma_{u_F}^2}{\gamma_D (1 + B_{F1}^2 \sigma_{u_F}^2)} \tag{7}
\]

\[
B_{F1} = f_{F1}(B_{D1}; \gamma_F, \sigma_{\eta_F}, d_F, \sigma_{u_D}) = \frac{\sigma_{\eta_F}^2}{\gamma_F} + \frac{d_F^2 B_{D1}^2 \sigma_{u_D}^2}{\gamma_F (1 + B_{D1}^2 \sigma_{u_D}^2)} \tag{8}
\]

Furthermore, \( R_{j1} = 1 \), and the coefficients, \( E_{j1} \), and \( C_{j1} \) can be expressed as functions of \( B_{j1} \) and \( B_{-j1} \) (See the appendix).

In contrast to the benchmark case, when \( \mu_D = \mu_F = 1 \), the illiquidities of securities \( D \) and \( F \) are interconnected: \( B_{D1} \) is a function of \( B_{F1} \) and vice versa if and only if \( d_D > 0 \) and \( d_F > 0 \). Thus, the number of equilibria is equal to the number of pairs \( \{B_{D1}, B_{F1}\} \) solving the system of equations (7) and (8). In general, we cannot characterize these solutions analytically and therefore cannot solve for the equilibria in closed-form. However, we can find these solutions numerically. The equilibria are the values of \( B_{D1} \) and \( B_{F1} \) at which the curves representing the functions \( f_{D1}(\cdot) \) and \( f_{F1}(\cdot) \) intersect, as Figure 2 shows for specific values of the parameters.\(^{18}\)

Multiple equilibria arise because liquidity and price informativeness are self-reinforcing in our model, as shown by Corollaries 1 and 2 below. The following lemma is useful to derive and understand these corollaries.

\(^{18}\)Proposition 1 focuses on the case in which \( \sigma_{\eta_D} > 0 \) or \( \sigma_{\eta_F} > 0 \). When \( \sigma_{\eta_D} = \sigma_{\eta_F} = 0 \), the result is slightly different because, in this case, dealers in both markets collectively hold all information on securities \( D \) and \( F \)’s payoffs. Cross-asset learning is a way to share this information and suppress all uncertainty about the final payoffs. Thus, when \( \sigma_{\eta_D} = \sigma_{\eta_F} = 0 \), there always exists a fully revealing equilibrium, where the price of each asset is equal to its final payoff and the market is infinitely deep: \( B_{D1} = B_{F1} = 0 \). Otherwise Proposition 1 is unchanged.
Lemma 2. When $\mu_D = \mu_F = 1$, in any equilibrium,

$$p_j^* = (1 - E_j1E_{-j1})\omega_j + E_j1p_{-j}^*, \quad \text{for } j \in \{F, D\}. \quad (9)$$

where $\omega_j \equiv \delta_j + B_{1j}u_j$ for $j \in \{D, F\}$. Hence, $\omega_{-j}$ is a sufficient statistic for the price information, $\mathcal{P}^W_j = \{p_j^*, p_{-j}^*\}$, available to pricewatchers operating in security $j$.

In other words, $\omega_{-j}$ is the signal about the risk factor $\delta_{-j}$ that dealers operating in security $j$ extract from the price of security $-j$. Hence the precision of the forecast formed by dealers in security $j$ about the payoff of this security is

$$\text{Var}[v_j|\delta_j, \omega_{-j}]^{-1} = \left(\text{Var}[v_j|\delta_j] \left(1 - \rho_{j1}\right)\right)^{-1}, \quad (10)$$

where

$$\rho_{j1} \overset{\text{def}}{=} \frac{E[v_j\omega_{-j}|\delta_j]^2}{\text{Var}[v_j|\delta_j]\text{Var}[\omega_{-j}]} . \quad (11)$$

Variable $\rho_{j1}$ measures the informativeness of the price of security $-j$ about the payoff of security $j$ for dealers operating in security $j$: The higher is $\rho_{j1}$, the greater is the precision of the signal conveyed by the price of security $-j$ to dealers in security $j$. Using the definition of $\omega_j$, we obtain

$$\rho_{j1} = \frac{d_j^2}{(\sigma^2_{\eta_j} + d_j^2)(1 + B_{-j1}^2\sigma^2_{u_j})} . \quad (12)$$

\footnote{Equation (10) follows from the fact that if $X$ and $Y$ are two random variables with normal distribution then $\text{Var}[X|Y] = \text{Var}[X] - \text{Cov}^2[X, Y]/\text{Var}[Y]$.}
We deduce from Proposition 1 and equation (12) that

$$B_{j1} = B_{j0}(1 - \rho_{j1}).$$

(13)

The next corollary follows.

**Corollary 1.** When $\mu_D = \mu_F = 1$, an increase in the informativeness of the price of security $-j$ for dealers in security $j$ makes security $j$ more liquid, i.e.,

$$\frac{\partial B_{j1}}{\partial \rho_{j1}} \leq 0.$$  

(14)

The intuition for this result is straightforward. By watching the price of another security, dealers learn information. Hence, they face less uncertainty about the payoff of the security in which they are active, which makes the liquidity of this security higher.

This relationship also works in the opposite direction: an increase in the liquidity of one security makes the price of this security more informative for dealers in other securities. Indeed, variations in the price of security $j$ are relatively less driven by variations in liquidity demands (noise) for this security when security $j$ is more liquid. More formally, remember that the signal about factor $\delta_j$ conveyed by the price of security $j$ to dealers in security $-j$ is $\omega_j = \delta_j + B_{j1} \times u_j$. Clearly, this signal is noisier when $B_{j1}$ is higher, which yields the following result.

**Corollary 2.** When $\mu_D = \mu_F = 1$, an increase in the illiquidity of security $j$ makes its price less informative for dealers in security $-j$:

$$\frac{\partial \rho_{-j1}}{\partial B_{j1}} \leq 0.$$  

(15)

Corollaries 1 and 2 explain why the levels of illiquidity for securities $D$ and $F$ are interconnected: the illiquidity of security $-j$ determines the informativeness of its price for dealers in security $j$ (Corollary 2) and as a result the illiquidity of security $j$ (Corollary 1).

Corollaries 1 and 2 imply that price informativeness and liquidity are self-reinforcing: an increase in the liquidity of one asset makes its price more informative, which makes the liquidity of the other asset higher, etc. This self-reinforcing loop explains why multiple equilibria are possible. Suppose that dealers expect other assets to be illiquid and therefore prices to be relatively uninformative. Correspondingly, they perceive their positions as being risky since they have little information on the payoff of the asset in which they make the market. As a result, liquidity is low in all assets and prices are not very informative, which validates dealers’ beliefs. Symmetrically, beliefs that other assets are very liquid and prices very informative can be self-fulfilling as well. Thus, for the same parameter values, various levels of liquidity and price informativeness can be sustained in equilibrium.\(^{20}\) The next result provides a sufficient condition for equilibrium uniqueness.

\(^{20}\)Ganguli and Yang (2009) consider a single security model of price formation similar to Grossman and Stiglitz (1980). They show that multiple rational expectations equilibria can arise when investors have private information about the asset payoff and the aggregate supply of the asset (see also Manzano and Vives (2011)). Here, the source of multiplicity is different since dealers have no supply information in our model.
Corollary 3. When $\mu_D = \mu_F = 1$, there is a unique equilibrium if

$$\max \left\{ \frac{\sigma^2_{\eta_D}(\gamma_F^2 + \sigma^4_{\eta_F} \sigma^2_{u_F})}{d^2_F(4 \gamma_F^2 + 3 \sigma^4_{\eta_F} \sigma^2_{u_F})}, \frac{\sigma^2_{\eta_F}(\gamma_D^2 + \sigma^4_{\eta_D} \sigma^2_{u_D})}{d^2_D(4 \gamma_D^2 + 3 \sigma^4_{\eta_D} \sigma^2_{u_D})} \right\} > 1. \quad (16)$$

The L.H.S of Condition (16) increases with $\sigma^2_{\eta_D}$ or $\sigma^2_{\eta_F}$. Thus, other things equal, a unique equilibrium is more likely when the idiosyncratic risk of either one of the two assets is higher. To see why, suppose that the idiosyncratic risk of asset $D$, $\sigma^2_{\eta_D}$, is large relative to $d_D$, its loading on the factor unknown to dealers in security $D$ (that is $\delta_F$). Thus, a large share of the uncertainty on the payoff of asset $D$ is idiosyncratic and, therefore, the information conveyed by the price of security $F$ about factor $\delta_F$ does not reduce much the uncertainty about the payoff of security $D$. For this reason, the belief of dealers in security $D$ about the liquidity of security $F$ plays a relatively minor role in the determination of the liquidity of security $D$ and therefore the equilibrium is unique.\footnote{For instance, consider the polar case in which $d_D = 0$. In this case, dealers in security $D$ have no information to learn from the price of security $F$. Thus, the illiquidity of security $D$ is uniquely pinned down by its “fundamentals” $\gamma_D$ and $\sigma_{\eta_D}$ and, as a result, the beliefs of dealers in security $F$ regarding the liquidity of security $D$ are uniquely defined as well (since dealers’ expectations about the illiquidity of the other security must be correct in equilibrium). Thus, the equilibrium is unique.}

When there are three equilibria, they can be ranked in terms of liquidity and price informativeness for all securities. To see this, let us denote by $L$, $M$ and $H$ the equilibrium with Low, Medium, and High illiquidity for, say, security $D$ and let $B^k_j$ be the illiquidity of security $j$ when the equilibrium of type $k \in \{L, M, H\}$ is obtained for security $D$. By definition $B^L_{D_1} < B^M_{D_1} < B^H_{D_1}$. Hence, Corollary 2 implies that the informativeness of the price of security $D$ is relatively high in the equilibrium of type $L$, medium in the equilibrium of type $M$, and relatively low in the equilibrium of type $H$. In turn, Corollary 1 implies that $B^L_{F_1} < B^M_{F_1} < B^H_{F_1}$. We deduce that $B^L_{D_1} < B^M_{D_1} < B^H_{D_1}$ if and only if $B^L_{F_1} < B^M_{F_1} < B^H_{F_1}$, as shown on Figure 2. That is, if illiquidity is relatively low in one security, it must also be relatively low in the other security. Similarly, if illiquidity is relatively high in one security, it must also be relatively high in the other security. Thus, a switch from a Low to a High (or Medium) illiquidity equilibrium in one security will affect all securities in the same way, as if illiquidity was contagious.

This is a manifestation of a more general property: when $\mu_D = \mu_F = 1$, an exogenous change in the illiquidity of one market (due for instance to an increase in dealers’ risk tolerance in this market) affects the illiquidity of the other market in the same direction. We call this effect a positive liquidity spillover.

3.3 Contagion and amplification: Liquidity spillovers and the illiquidity multiplier

To see how liquidity spillovers arise, consider an exogenous decrease in the illiquidity of security $D$ (e.g., due to an increase in the risk tolerance of dealers specialized in this asset). Its price then becomes more informative for dealers in security $F$ (Corollary 2), which in turn becomes more
liquid (Corollary 1). Thus, in contrast to the benchmark case, the improvement in the liquidity of security $D$ spreads to security $F$ although security $F$ experiences no change in its liquidity fundamentals.

In order to formalize this point, consider the system of equations (7) and (8). Any exogenous change in the illiquidity of security $j$ will spill over to security $-j$ when $\frac{\partial f_{-j1}}{\partial B_{j1}} \neq 0$, that is, when $d_{-j} > 0$. The direction (positive/negative) of this spillover is determined by the sign of $\frac{\partial f_{-j1}}{\partial B_{j1}}$. The next corollary shows that when $\mu_D = \mu_F = 1$, this sign is always positive: an increase in the illiquidity of one asset makes the other asset more illiquid as well.

**Corollary 4.** When $\mu_D = \mu_F = 1$ and $d_j > 0$, liquidity spillovers from security $-j$ to security $j$ are always positive (e.g., $\frac{\partial f_{j1}}{\partial B_{-j1}} > 0$ when $d_j > 0$), for $j \in \{D, F\}$. When $d_j = 0$, there is no spillover from security $-j$ to security $j$ ($\frac{\partial f_{j1}}{\partial B_{-j1}} = 0$) because the price of security $-j$ conveys no information to dealers in security $j$.

When illiquidity spillovers operate in both directions (that is, $d_j > 0$), a small shock to one liquidity fundamental is amplified through feedback effects. Consider for instance the chain of effects that follows a small reduction, denoted by $\Delta \gamma_D < 0$, in the risk tolerance of dealers in security $D$. The direct effect of this reduction is to increase the illiquidity of security $D$ by $(\partial f_{D1}/\partial \gamma_D)\Delta \gamma_D > 0$. Hence, the price of this security becomes less informative for dealers in security $F$. When $d_F > 0$, these dealers face therefore more uncertainty so that security $F$ becomes less liquid as well, although its liquidity fundamentals are unchanged. The immediate increase in the illiquidity of security $F$ is equal to $(\partial f_{F1}/\partial B_{D1})(\partial f_{D1}/\partial \gamma_D)\Delta \gamma_D > 0$. As a result, the illiquidity of security $D$ increases even more, starting a new vicious feedback loop: the increase in illiquidity for security $D$ leads to a further increase in illiquidity for security $F$, etc. In sum, the total effect of the initial decrease in the risk tolerance of dealers in security $D$ is an order of magnitude larger than its direct effect on the illiquidity of both securities.

More generally, the next corollary shows that the direct effects of a change in any liquidity fundamental of a given asset are amplified by a factor $\kappa > 1$ when dealers in each asset learn from each other prices (that is, $d_j > 0$). We refer to $\kappa$ as the illiquidity multiplier.

**Corollary 5.** Suppose $\mu_D = \mu_F = 1$, and let

$$
\kappa(B_{D1}, B_{F1}) \equiv \left(1 - \frac{\partial f_{F1}}{\partial B_{D1}} \frac{\partial f_{D1}}{\partial B_{F1}}\right)^{-1}.
$$

(17)

Suppose that parameters are such that the equilibrium is unique and let $B^*_j$ be the level of illiquidity in this equilibrium for security $j$. Then, $\kappa(B^*_{D1}, B^*_{F1}) > 1$ when $d_j > 0$ for $j \in \{D, F\}$ and $\kappa(B^*_{D1}, B^*_{F1}) = 1$ when $d_D = 0$ or $d_F = 0$. Moreover, a small increase in one liquidity fundamental
of security \( j \), \( \chi_j \in \{ \gamma_j, \sigma_{nj}, \text{or } d_j \} \), changes the illiquidity of securities \( j \) and \( -j \) by

\[
\begin{align*}
\frac{dB_{j1}}{d\chi_j} & = \frac{\partial f_{j1}}{\partial B_{j1}} \frac{b_{j1} - B_{j1}^*}{B_{-j1}^* - b_{-j1}^*}, \\
\frac{dB_{-j1}}{d\chi_j} & = \frac{\partial f_{-j1}}{\partial B_{-j1}} \frac{b_{-j1} - B_{-j1}^*}{B_{-j1}^* - b_{-j1}^*}.
\end{align*}
\]

Total Effect

Thus, idiosyncratic shocks to the illiquidity of one asset translate into positive co-movements in the illiquidity of both assets. For instance, a decrease in the risk tolerance of dealers in asset \( D \) ultimately triggers an increase in the illiquidity of both assets \( D \) and \( F \) (\( \frac{dB_{D1}}{d\gamma_D} > 0 \) and \( \frac{dB_{F1}}{d\gamma_D} > 0 \)), although the risk tolerance of dealers in asset \( D \) has no direct effect on the illiquidity of asset \( F \).

Corollary 5 focuses on the case in which there is a unique equilibrium. When there are multiple equilibria, Corollary 5 remains valid for the two extreme equilibria: the equilibrium in which the illiquidity of both securities is relatively Low and the equilibrium in which the illiquidity of both securities is relatively High (see the proof of the corollary). The equilibrium with a Medium level of illiquidity is such that \( \kappa < 0 \) and delivers “counter-intuitive” comparative statics. For instance, in this equilibrium, a reduction in the risk tolerance of dealers in, say, security \( D \) reduces the liquidity of both securities. Such an equilibrium may exist because, as explained previously, the illiquidity of each security is in part determined by dealers’ beliefs about the illiquidity of the other market. These beliefs may be disconnected from the illiquidity fundamentals of each security and yet be self-fulfilling.

This point is closely linked to the notion of equilibrium stability. In our setting, an equilibrium can be viewed as stable if, when we shock the illiquidity of one of the two securities by a small amount and trace the evolution of illiquidity using the system of equations (7) and (8), then we are brought back to the same equilibrium point. It can be shown that stability in this sense is obtained if and only if \( \kappa(B_{D1}^*, B_{F1}^*) \geq 1 \). Thus, only the two extreme equilibria are stable. For this reason, when the model has multiple equilibria, we focus on those for which \( \kappa \geq 1 \) (this is automatically the case if the equilibrium is unique).

\[\text{(For instance, suppose that the Low illiquidity equilibrium is obtained and increase the illiquidity of security } D \text{ from } B_{D1}^L \text{ to } B_{D1}^L + \Delta \text{ where } \Delta \text{ is small. Then using the system of equations (7) and (8), we can compute the new value of } B_{F1} \text{ and then compute the resulting value of } B_{D1} \text{ and so on and so forth. If, following this process, the illiquidity of both markets converges back to } (B_{D1}^L, B_{F1}^L) \text{ then the Low illiquidity equilibrium is stable (in Figure 2 we carry out this exercise for the equilibrium point } M, \text{ showing that it is unstable). If one views the system of equations (7) and (8) as a system of reaction functions, our notion of stability and the resulting condition on the slopes of the reaction functions are standard in game theory (see Fundenberg and Tirole (1991), Chapter 1, Section 1.2.5).}\]
3.4 Liquidity crashes, liquidity spikes and the size of the illiquidity multiplier.

A price crash is often described as a situation in which a small change in economic conditions is associated with an overly large drop in asset prices. For instance, Yuan (2005) defines an asset market crisis as “a large price drop in response to a small shock to the economic environment” and for Barlevy and Veronesi (2003), the main feature of a crash is that: “a small change in the underlying fundamentals is associated with a disproportionately large change in the price of the asset.” By analogy, we define a liquidity crash as a situation in which a small change in the illiquidity fundamental of one asset triggers a very large increase in the illiquidity of all assets.

As our model is static, we can only analyze the effects of a change in the economic environment through comparative statics, that is, by comparing equilibrium levels of illiquidity for small variations of the parameters. This approach follows that used in other static models of market crashes (e.g., Genotte and Leland (1990), Barlevy and Veronesi (2003), Yuan (2005), or Breon-Drish (2010)).

In our model, liquidity crashes can happen when the illiquidity multiplier is high. To see this, consider Table 1. It gives the elasticity \( E_{\gamma_j} \) of the illiquidity of security \( j \) to the risk tolerance of dealers in security \( D \), that is, the percentage change in the illiquidity of each security for a one percent increase in the risk tolerance of dealers in security \( D \). The table also gives the same elasticities \( \hat{E}_{\gamma_j} \) in an economy without feedback (that is, when we set \( \kappa = 1 \)).

Table 1 shows two things. First, in equilibrium, the illiquidity of each asset is more sensitive to a shock to dealers’ tolerance in asset \( D \) than it would in the absence of feedback. This is due to the amplification effect described in Corollary 5. Second, there exist cases in which the amplification effect is very large, even for very small shocks to dealers’ risk tolerance. For instance when \( \gamma_D = 1 \), the illiquidity multiplier is very high (\( \kappa = 1,779 \)). In this case, a one percent drop in the risk tolerance of dealers in security \( D \) triggers an increase of 30% and 29% in the illiquidity of securities \( D \) and \( F \), respectively, versus only 1% and 2.15% in the absence of feedback effects.

This case is by no means special. In fact, as shown by the next corollary, there always exists parameter values such that the illiquidity multiplier is very large.

**Corollary 6.** Suppose that \( \sigma_{\eta_j} = \sigma_\eta \), \( \gamma_j = \gamma \), \( \sigma_{u-j} = \sigma_u \), and \( d_j = d < d^c \), where \( d^c = 2\sqrt{2}\sigma_\eta \). Then:

1. The illiquidity multiplier is maximal when dealers’ risk tolerance is \( \hat{\gamma} = (\sqrt{3}\sigma_u(d^2 + 4\sigma_\eta^2))/4 \).
equilibrium is unstable. Thus, the three equilibria.

For both assets. For instance, in our previous numerical example, a one percent increase in the aversion of dealers in market $D$ is maximal for $\gamma_D = 1$, yielding $\kappa(1) = 1,779$.

Table 1: The table shows the impact of the illiquidity multiplier for different shocks to the risk aversion of dealers in market $D$. Other parameter values are $d_D = d_F = 1$, $\sigma_{u_F} = 1$, $\sigma_{u_D} = 1.5$, $\gamma_F = .6328$, $\sigma_{\eta_D} = .36728$, and $\sigma_{\eta_F} = .339638$. In this case, $\kappa$ peaks at $\gamma_D = 1$, yielding $\kappa = 1,779$.

<table>
<thead>
<tr>
<th>$\gamma_D$</th>
<th>$\kappa$</th>
<th>$B_{D1}$</th>
<th>$B_{F1}$</th>
<th>$\hat{E}<em>{B</em>{D1},\gamma_D}$</th>
<th>$\hat{E}<em>{B</em>{F1},\gamma_D}$</th>
<th>$\hat{E}<em>{B</em>{D1},\gamma_D}$</th>
<th>$\hat{E}<em>{B</em>{F1},\gamma_D}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.02</td>
<td>6.11</td>
<td>0.27</td>
<td>0.41</td>
<td>-6.11</td>
<td>-1.00</td>
<td>-18.93</td>
<td>-3.1</td>
</tr>
<tr>
<td>1.01</td>
<td>9.42</td>
<td>0.29</td>
<td>0.44</td>
<td>-9.42</td>
<td>-1.00</td>
<td>-27.03</td>
<td>-2.87</td>
</tr>
<tr>
<td>1</td>
<td>1,779</td>
<td>0.39</td>
<td>0.59</td>
<td>-1,779</td>
<td>-1.00</td>
<td>-3,826</td>
<td>-2.15</td>
</tr>
<tr>
<td>0.99</td>
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<td>0.51</td>
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<td>-9.37</td>
<td>-1.00</td>
<td>-15.64</td>
<td>-1.67</td>
</tr>
<tr>
<td>0.98</td>
<td>6.05</td>
<td>0.54</td>
<td>0.82</td>
<td>-6.05</td>
<td>-1.00</td>
<td>-9.33</td>
<td>-1.54</td>
</tr>
<tr>
<td>0.97</td>
<td>4.71</td>
<td>0.58</td>
<td>0.86</td>
<td>-4.71</td>
<td>-1.00</td>
<td>-5.85</td>
<td>-1.45</td>
</tr>
<tr>
<td>.96</td>
<td>3.96</td>
<td>0.60</td>
<td>0.89</td>
<td>-3.96</td>
<td>-1.00</td>
<td>-5.48</td>
<td>-1.38</td>
</tr>
<tr>
<td>.95</td>
<td>3.47</td>
<td>0.63</td>
<td>0.92</td>
<td>-3.47</td>
<td>-1.00</td>
<td>-4.59</td>
<td>-1.32</td>
</tr>
</tbody>
</table>

in which case the value of the illiquidity multiplier is equal to

$$\kappa^{\max} = \left(1 - \frac{9d^4}{4(4\sigma_u^2 + d^2)^2}\right)^{-1},$$

and the illiquidity of each asset is equal to $(\sqrt{3}\sigma_u)^{-1}$.

2. The illiquidity multiplier, $\kappa^{\max}$ increases in $d$ and goes to infinity when $d$ goes to $d^*$ from below \((\lim_{d \to (d^*)^-} \kappa^{\max} = \infty)\).\(^{26}\)

3. Hence, there always exist values of the parameters such that the illiquidity multiplier is higher than any arbitrary threshold, $\bar{\kappa}$, in equilibrium.

A large illiquidity multiplier implies that a small increase in the illiquidity of one asset triggers a very large increase in the illiquidity of all assets. By symmetry, in this situation, a small decrease in the illiquidity of an asset generates a liquidity spike, that is, a large positive jump in liquidity for both assets. For instance, in our previous numerical example, a one percent increase in the

\(^{26}\)It is easily shown that when parameters are identical for both assets, illiquidity is identical for both assets in each equilibrium. Moreover, in this case, it is easily shown that if $B^*$ is the illiquidity of each asset in a stable equilibrium then $\kappa$ is maximal for $B^* = B^{\max}$ where $B^{\max} = (\sqrt{3}\sigma_u)^{-1}$. To obtain Corollary 6, we choose the value for $\gamma$ such that $B^* = B^{\max}$ is a possible equilibrium. This value is $\hat{\gamma}$. The condition $d < d^*$ guarantees that this equilibrium is stable. If $d > d^*$, the equilibrium in which $B^* = B^{\max}$ is still an equilibrium when $\gamma = \hat{\gamma}$ but this equilibrium is unstable. Thus, $\kappa^{max}$ becomes negative and $\kappa^{\max}$ is not the largest but the smallest value of $\kappa$ across the three equilibria.
risk tolerance of dealers in asset $D$ triggers a decrease in the illiquidity of assets $D$ and $F$ of about twenty five percent. Thus, our model implies either liquidity crashes or liquidity spikes. An interesting empirical question is whether the distribution of changes in liquidity is symmetric (as implied by our model) or negatively skewed. We are not aware of empirical work on this question.

There also exist parameters for which $\kappa$ is small and close to 1, as Table 1 shows. More generally, numerical simulations show that the illiquidity multiplier is low for relatively small or large values of the parameters and peaks for intermediate values. Figure 3 illustrates this pattern by plotting the (log of the) illiquidity multiplier, $\kappa$, against the idiosyncratic risk of asset $D$, $\sigma_{\eta_D}$ (panel (a)) and dealers’ risk tolerance, $\gamma_D$ (panel (b)).

![Figure 3: The illiquidity multiplier. Panel (a) displays $\ln \kappa(\sigma_{\eta_D})$, where $\sigma_{\eta_D} \in \{.0001,.0011,\ldots,5\}$, and the other parameter values are as follows: $d_D = d_F = 1$, $\sigma_u = 1$, $\sigma_{\eta_D} = 1.5$, $\gamma_D = 1$, $\gamma_F = .6328$, and $\sigma_{\eta_F} = .339638$. In this case $\kappa$ peaks for $\sigma_{\eta_D} = .36728$, yielding $\kappa(.36728) = 1.779$. Panel (b) displays $\ln \kappa(\gamma_D)$, where $\gamma_D \in \{.001,.002,\ldots,1.5\}$, and the other parameter values are as follows: $d_D = d_F = 1$, $\sigma_u = 1$, $\sigma_{\eta_D} = 1.5$, $\gamma_F = .6328$, $\sigma_{\eta_D} = .36728$, and $\sigma_{\eta_F} = .339638$. In this case, $\kappa$ peaks at $\gamma_D = 1$, yielding $\kappa(1) = 1.779$.](image)

The reason for this behavior is as follows. The size of the illiquidity multiplier is large when the product of the sensitivity of the illiquidity of each market to the illiquidity of the other market ($\partial f_F^1/\partial B_{D1} \times \partial f_{D1}/\partial B_F$) is large (see Equation (17)). Using equations (7) and (8), we obtain that the sensitivity of the illiquidity of asset $j$ to the illiquidity of asset $-j$ is

$$
\frac{\partial f_{j1}}{\partial B_{-j1}} = \left(\frac{2d_j^2\sigma_{u_j}^2}{\gamma_J}\right) \left(\frac{B_{-j1}}{1 + B_{-j1}^2\sigma_{u_{-j}}^2}\right)^2.
$$

This sensitivity is a concave function of the illiquidity of asset $-j$, reaching a maximum when $B_{-j1} = \left(\sqrt{3}\sigma_{u_{-j}}\right)^{-1}$.

27Differentiating equation (18) we obtain

$$
\frac{\partial^2 f_{j1}}{\partial^2 B_{-j1}} = \left(\frac{2d_j^2\sigma_{u_j}^2}{\gamma_J}\right) \left(\frac{1 - 3B_{-j1}^2\sigma_{u_{-j}}^2}{(1 + B_{-j1}^2\sigma_{u_{-j}}^2)^3}\right) \Rightarrow \frac{\partial^j f_{j1}}{\partial^j B_{-j1}} < 0
$$

. Thus, $(\partial f_{j1}/\partial B_{-j1})$ is concave.
illiquidity in each market are close to \((\sqrt{3} \sigma_{u_j})^{-1}\) and small otherwise. In equilibrium, illiquidity levels in both markets are monotonic in the parameters (e.g., they decrease in \(\gamma_j\) and increase in \(\sigma_{\eta_j}\)). Consequently, the illiquidity multiplier tends to be low for relatively low or large values of the parameters.

The sensitivity of the illiquidity of each market to the illiquidity of the other market \((\partial f_{j1}/\partial B_{-j1})\) plays an important role in the model. These sensitivities can be estimated by regressing the illiquidity of an asset at a given date on lagged levels of illiquidity for other assets, controlling for other characteristics affecting liquidity (e.g., idiosyncratic risk and factor loadings for each asset according to our model). This approach could be used to assess the size and direction of liquidity spillovers from one asset to another and to get an estimate of the illiquidity multiplier.

Early warnings indicators for the illiquidity multiplier would be useful because liquidity meltdowns are associated with a significant increase in the steepness of “excess demand curves” in each asset (i.e., the function relating prices to liquidity traders’ demands). Thus, when the illiquidity multiplier is high, a small decrease in dealers’ risk tolerance in one asset can lead to a sharp increase in price volatility for all assets.

![Figure 4](image)

Figure 4: The effect of an increase in dealers’ risk aversion on the volatility of asset prices. We assume \(d_D = d_F = 1\), \(\sigma_{u_F} = 1\), \(\sigma_{u_D} = 1.5\), \(\gamma_F = .6328\), \(\sigma_{\eta_D} = .36728\), and \(\sigma_{\eta_F} = .339638\). In this case, \(\kappa\) peaks at \(\gamma_D = 1\), yielding \(\kappa(1) = 1.779\).

As an illustration, Figure 4 shows the effect of an increase in risk aversion on the volatility of asset prices when the market approaches the set of parameter values for which \(\kappa\) spikes. We choose the same parameter values used in Table 1, for which we know that \(\kappa = 1.779\) when \(\gamma_D = 1\). As shown by the figure, the volatilities of the two asset prices are substantially stable for \(\gamma_D > 1\), but start rapidly increasing once \(\gamma_D\) becomes lower than 1.

\(^{28}\)Empirically, the challenge is to account for unobservable variables that can affect time-variations in liquidity of all assets. This problem is conceptually identical to that of identifying peer effects (see Leary and Roberts (2010) for a discussion in a corporate finance context).
In our model, extreme variations in liquidity for small changes in the economic environment can happen whether the equilibrium is unique or not (in all our numerical examples in this section, the equilibrium is unique). When there are multiple equilibria, liquidity crashes might also happen when market participants switch from the low illiquidity equilibrium to the high illiquidity equilibrium. This mechanism for liquidity crashes however leaves unexplained the cause of the switch. This problem does not arise when a liquidity crash is due to a high illiquidity multiplier: in this case, a small change in the economic environment, such as for instance a small decrease in dealers’ risk tolerance, is the origin of the sudden increase in illiquidity.

4 Extensions

In this section, we extend the model in two directions. First, in Section 4.1 we relax the assumption that all dealers in each asset observe the price of the other asset. Second, in Section 4.2 we introduce cross-market arbitrageurs who act as liquidity providers in both assets and study how these arbitrageurs affect the illiquidity multiplier.

4.1 Illiquidity spillovers and the scope of access to price information

We now generalize the baseline model for arbitrary fractions, \( \mu_D \) and \( \mu_F \), of pricewatchers in each market. As when \( \mu_D = \mu_F = 1 \), a linear rational expectations equilibrium is a set of prices \( \{p^*_j\}_{j \in \{D,F\}} \) such that

\[
p^*_j = R_j \delta_j + B_j u_j + E_j \delta_{-j} + C_j u_{-j},
\]

and \( p^*_j \) clears the market of asset \( j \) for each realization of \( \{u_j, \delta_j, u_{-j}, \delta_{-j}\} \) when dealers anticipate that clearing prices satisfy equation (19) and choose their trading strategies to maximize their expected utility. The next proposition extends the existence result in Proposition 1 for arbitrary values of \( \mu_D \) and \( \mu_F \).

**Proposition 2.** For any value of \( \mu_D \) and \( \mu_F \), there always exists a non-fully revealing linear rational expectations equilibrium. At equilibrium, \( B_j > 0, R_j = 1 \) and the other coefficients (\( E_j \) and \( C_j \)) can be expressed as functions of \( B_j \) and \( B_{-j} \). Moreover,

\[
B_j = B_{j0} (1 - \rho_j) \times \frac{\gamma^2_j \mu_j \rho_j + \sigma^2_{u_j} \text{Var}[v_j|\delta_j](1 - \rho_j)}{\gamma^2_j \mu^2_j \rho^2_j + \sigma^2_{u_j} \text{Var}[v_j|\delta_j](1 - \rho_j)(1 - \rho_j(1 - \mu_j))}, \text{ for } j \in \{D, F\}
\]

where \( \rho_j \equiv d^2_j / ((\sigma^2_{u_j} + d^2_j)(1 + B^2_{-j} \sigma^2_{u_{-j}})) \).

---

29 Exchange rate crises have often been interpreted as the result of a jump from one equilibrium to another in models with multiple equilibria (see for instance Obstfeld (1986)).

30 Bernhardt and Taub (2008) consider a multi-asset model of trading similar to Caballé and Krishnan (1994) and analyze the effect of price observability in this model. The structure of their model is quite different, however (e.g., dealers in each market have the same information, are risk neutral etc.). In their model, there is a unique linear rational expectations equilibrium.
As in the baseline case, pricewatchers in security $j$ extract a signal $\omega_j = \delta_j - j + B_j u_j$ from the price of security $-j$ and $\rho_j$ is the informativeness of this signal. As pricewatchers’ trading strategy depends on their signal (i.e., $\omega_j$), the price of security $j$ partially reveals pricewatchers’ private information to inattentive dealers. Equation (19) implies that observing the price of security $j$ and risk factor $\delta_j$ is informationally equivalent to observing $\hat{\omega}_j \equiv E_j \delta_j - j + C_j u_j + B_j u_j$. Thus, in equilibrium, the information set of inattentive dealers, $\{\delta_j, p_j\}$, is informationally equivalent to $\{\delta_j, \hat{\omega}_j\}$. We refer to $\hat{\omega}_j$ as inattentive dealers’ price signal. In the proof of Proposition 2, we show that $\hat{\omega}_j = E_j \omega_j + B_j u_j$. Accordingly, when $B_j > 0$, inattentive dealers in security $j$ are at an informational disadvantage compared to pricewatchers because signal $\hat{\omega}_j$ is less precise than $\omega_j$. This disadvantage exposes inattentive dealers to adverse selection.

Substituting $\rho_D$ and $\rho_F$ by their expressions in equation (20), we can express $B_j$ as a function of $B_{-j}$. We obtain:

$$B_D = f_D(B_F; \mu_D, \gamma_D, \sigma_u^2, d_D, \sigma_u D), \quad (21)$$

$$B_F = f_F(B_D; \mu_F, \gamma_F, \sigma_u^2, d_F, \sigma_u F), \quad (22)$$

where the expression for $f_j(\cdot)$ is given in the Appendix (see equation (A.17)). The linear rational expectations equilibria are completely characterized by the solution(s) of this system of equations. Numerical simulations show that there can be either one or three non-fully revealing equilibria as in the baseline case.\(^{31}\)

In any case, liquidity spillovers arise as long as the fraction of pricewatchers is not zero in both markets. However, in contrast to the baseline case, liquidity spillovers are not necessarily positive when $\mu_j < 1$. That is, there might exist cases in which an increase in the illiquidity of one asset reduces the illiquidity of the other asset (that is, $\partial f_j / \partial B_{-j} < 0$). To see why, consider a decrease in the risk tolerance of dealers operating in asset $D$ ($\gamma_D$ decreases) and suppose that $\mu_F < 1$. Asset $D$ becomes less liquid and therefore less informative for pricewatchers in asset $F$. Thus, uncertainty about the payoff of asset $F$ increases. This “uncertainty effect” increases the illiquidity of asset $F$ as when $\mu_F = 1$. However, when $\mu_F < 1$, there is a countervailing “adverse selection effect” because as pricewatchers’ information about the price of asset $D$ becomes less precise, their informational advantage is smaller. Correspondingly, inattentive dealers in asset $F$ are less exposed to adverse selection. This effect reduces the illiquidity of asset $F$. The net effect of an increase in the illiquidity of asset $D$ on the illiquidity of asset $F$ is positive only if the uncertainty effect dominates the adverse selection effect.

The next corollary provides a sufficient condition for this to be the case. For this corollary, we

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\(^{31}\)As explained in the previous section, multiple equilibria arise in our model when dealers in one market rely strongly on the price of the other market as a source of information. This suggests that a necessary condition for the existence of multiple equilibria in our model is that the fractions of pricewatchers in both markets, $\mu_D$ and $\mu_F$, are high enough. We have not been able to prove this claim analytically. However extensive numerical simulations indicate that it holds true.
define
\[ R_j = \frac{\gamma_j^2}{\sigma_{u_j}^2 \text{Var}[v_j|\delta_j]}, \] (23)
which is a measure of the risk bearing capacity of dealers in asset \( j \).\footnote{In equilibrium, the aggregate inventory position of dealers in security \( j \) after trading at date 2 is \(-u_j\) and the total dollar value of this position at date 1 is \(-u_j \times v_j\). The risk associated with this position for inattentive dealers in security \( j \) is measured by its variance conditional on information on risk factor \( \delta_j \), i.e., \( \sigma_{u_j}^2 \text{Var}[v_j|\delta_j] \). Thus, \( R_j \) measures dealers’ risk tolerance per unit of risk taken by the dealers in the aggregate.}

**Corollary 7.** Let
\[ \bar{\mu}_j = \max \left\{ 0, \frac{R_j - 1}{R_j} \right\}, \text{ for } j \in \{D, F\}. \] (24)
If \( \mu_j \in [\bar{\mu}_j, 1] \) then liquidity spillovers from security \(-j\) to security \( j \) are positive (that is, an increase in the illiquidity of security \(-j\) makes security \( j \) more illiquid).

When dealers’ risk bearing capacity is not too large (\( R_j \leq 1 \) for \( j \in \{D, F\} \)), the uncertainty effect prevails for all values of \( \mu_j \) (i.e., \( \bar{\mu}_j = 0 \)) because risk sharing considerations are first order relative to informational asymmetries among dealers. In this case, liquidity spillovers are positive. Otherwise (\( R_j > 1 \) for \( j = D \) or \( j = F \)), liquidity spillovers are positive only if the fraction of pricewatchers in both markets is high enough since then \( \bar{\mu}_D > 0 \) or/and \( \bar{\mu}_F > 0 \).

As explained previously, positive liquidity spillovers translate into positive co-movements in liquidity for both assets. Intuitively, the size of the co-movements induced by a specific type of shocks (e.g., shocks to the risk tolerance of dealers in asset \( D \)) should increase with the fraction of pricewatchers in either markets because liquidity spillovers in our model stems from inferences that dealers make from the prices of other assets. Thus, if more dealers observe prices, co-movements should be stronger.

We illustrate this testable implication of the model with the following experiment. For a given value of \( \mu_F \), we compute the illiquidity of securities \( D \) and \( F \) assuming that \( \gamma_D \) is uniformly distributed in \([0.5, 1]\) for \( \sigma_{u_F} = \sigma_{u_D} = 1/2 \), \( d_F = 1 \), \( d_D \in \{0, 1\} \), \( \sigma_{\eta_D} = 2 \), \( \sigma_{\eta_F} = 0 \), \( \gamma_F = 1/2 \), and \( \mu_D \in \{0.1, 1\} \). For these values of the parameters \( \bar{\mu}_j = 0 \) and liquidity spillovers are therefore positive. We then compute the covariance between the resulting equilibrium values for \( B_D \) and \( B_F \).

Figure 5, Panel (a) shows this covariance as a function of \( \mu_F \) when \( d_D = 0 \), and \( \mu_D = 0.1 \), while in Panel (b) we assume that \( d_D = 1 \) and \( \mu_D \in \{0.1, 1\} \). In both cases, the covariance between the illiquidity of securities \( D \) and \( F \) is positive because illiquidity spillovers are positive. Moreover, as expected, covariations in illiquidity increases in the fraction of pricewatchers. As explained in the introduction, one could use the introduction of new technologies facilitating the dissemination of price information across securities to test whether an increase in the fraction of pricewatchers has indeed a positive causal effect on co-movements in illiquidity.
Figure 5: Comovement in illiquidity. The figure displays the covariance between the illiquidity of security $F$ and the illiquidity of security $D$ as a function of $\mu_F$ when $d_D = 0$, $d_F = 1$, $\mu_D = .1$, $\sigma_{\eta_D} = 2$, $\sigma_{\eta_F} = 0$, $\sigma_{u_D} = \sigma_{u_F} = 1/2$, and $\gamma_F = 1/2$ (panel (a)) and $d_D = d_F = 1$, $\sigma_{\eta_D} = 2$, $\sigma_{\eta_F} = 0$, $\sigma_{u_D} = \sigma_{u_F} = 1/2$, and $\gamma_F = 1/2$ (panel (b)). In panel (b) the covariance between the illiquidity of the two securities is higher when $\mu_D = 1$ (bold curve) than when $\mu_D = 0.1$ (dotted curve), for all values of $\mu_F > 0$.

Interestingly, an increase in the fraction of pricewatchers is itself a source of illiquidity spillovers, as shown by the next corollary. As the proof of this result is similar to that of Corollary 5, we provide it in the Internet Appendix for brevity.\footnote{As for Corollary 5, Corollary 8 holds for the two stable equilibria when there are multiple equilibria.}

**Corollary 8.** Suppose $R_j \leq 1$ for $j \in \{D, F\}$. Let $\kappa$ be the illiquidity multiplier defined in Corollary 5. A small increase in the fraction of pricewatchers in security $j$ decreases the illiquidity of both securities in equilibrium by $dB_j/d\mu_j = \kappa \times (\partial f_j/\partial \mu_j) < 0$ and $dB_{-j}/d\mu_j = \kappa \times (\partial f_{-j}/\partial B_j)(\partial f_j/\partial \mu_j) < 0$.

Corollary 8 implies that when $R_j \leq 1$, an increase in the fraction of pricewatchers in either market makes both securities more liquid. This result has the following testable implication. Consider a change in market organization that facilitates the dissemination of information on the price of a set of assets, playing the role of asset $F$. This change should therefore enable more dealers in other assets, playing the role of asset $D$, to obtain information on the price of asset $F$. This corresponds to an increase in $\mu_D$ in our model. Thus, a wider dissemination of prices for a set of assets should improve the liquidity of other assets with correlated payoffs (since $dB_D/d\mu_D < 0$), even if market structure for the latter set of assets is unchanged. In turn, this improvement should in itself improve the liquidity of the assets directly affected by the change in market structure (since $dB_F/d\mu_D > 0$).

In 2002, the National Association of Securities Dealers (NASD) began to report transaction prices in a set of corporate bonds, using a reporting system called TRACE. In line with our predictions, Bessembinder et al. (2006) show that the initiation of the TRACE reporting system in
the U.S. bond market improved the liquidity of bonds that were not eligible for TRACE. It also improved the liquidity of bonds eligible for TRACE. According to our model, this effect could be due in part to the improvement in liquidity of ineligible bonds.

4.2 Illiquidity spillovers with cross-market arbitrageurs

In our model, dealers only trade the asset for which they have private information. This assumption is convenient to turn off cross-market hedging effects and thereby highlight the role of cross-asset learning in liquidity spillovers. A natural question however is how the positive feedback loop between the liquidity of assets $D$ and $F$ is affected when there are liquidity providers active in both markets. To address this question, we introduce “cross-market arbitrageurs;” traders who have no direct risk-factor information on assets $D$ and $F$ but who can take a position in each asset. For tractability, we maintain the assumption that dealers with private information on risk factors are specialized. Otherwise, dealers operating in a given asset would have non nested information sets.\footnote{Dealers in asset $j$ have perfect information on $\delta_j$ and imperfect information on $\delta_{-j}$.} This feature greatly complicates the analysis without adding economic insights since cross-market arbitrageurs are sufficient to capture potential effects coming from multi-market trading.

Cross-market arbitrageurs in our model act as dealers: they absorb excess demand from liquidity traders. For instance, if liquidity traders sell $u_F$ shares in security $F$, these shares are purchased by dealers and cross-market arbitrageurs in equilibrium. We use the term “cross-market arbitrageurs” rather than “cross-market dealers” since liquidity suppliers using cross-market trading strategies are often referred to “arbitrageurs” (see CFTC-SEC (2011)). This interpretation is also standard in the literature on limits to arbitrage (see Gromb and Vayanos (2010) for a survey).

Each arbitrageur has a CARA utility function with risk tolerance $\gamma_A$. Hence, if arbitrageur $i$ holds $x_{ij}$ shares of asset $j$, her expected utility is

$$E[U(\pi_i) | \{p_j, p_{-j}\}] = E[- \exp \{-\gamma_A^{-1} \pi_i\} | p_j, p_{-j}], \quad (25)$$

where $\pi_i = \sum_{j \in \{D,F\}} (v_j - p_j)x_{ij}$. We denote by $x^A_j(p_j, p_{-j})$ the optimal position of a cross-market arbitrageur in asset $j$. It depends on the prices of each asset since (i) cross-market arbitrageurs, like dealers, can submit price-contingent orders and (ii) as usual in rational expectations models, they account for the information conveyed by prices in determining their optimal strategies.

In the presence of cross-market arbitrageurs, the clearing price of security $j$ is such that

$$\mu_j x^W_j(\delta_j, p_j, p_{-j}) + (1 - \mu_j)x^I_j(\delta_j, p_j) + \lambda x^A_j(p_j, p_{-j}) + u_j = 0, \quad \text{for } j \in \{D, F\}, \quad (26)$$

where $\lambda$ is the mass of arbitrageurs relative to the mass of dealers. Thus, $\lambda$ is an index of the amount of capital committed to cross-market arbitrage.
4.2.1 A benchmark: no specialized dealers

To better isolate the role of cross-market arbitrageurs, we first solve for the equilibrium in the absence of dealers. In this particular case, the equilibrium is unique (see Appendix B). Equilibrium prices are:

\[ p_j = \left( \frac{1 + d_j^2 + \sigma^2_{\eta_j}}{\lambda \gamma_A} \right) u_j + \left( \frac{d_D + d_F}{\lambda \gamma_A} \right) u_{-j}, \text{ for } j \in \{D, F\}. \] (27)

Cross-market arbitrageurs propagate the price pressure due to a liquidity demand shock in one asset to the other asset. Consider for instance a sell order in security \( F \) (\( u_F < 0 \)), holding the liquidity demand for security \( D \) at its mean level (\( u_D = 0 \)). The price of security \( F \) must drop to induce cross-market arbitrageurs to take a long position in security \( F \). Moreover, if the price of security \( D \) were unchanged, cross-market arbitrageurs would sell security \( D \) to hedge their position. Thus, the price of security \( D \) must drop as well in equilibrium since, in the absence of dealers, there is no trade in asset \( D \) if \( u_D = 0 \).

The baseline model with no cross-market arbitrageurs delivers a similar prediction but for a different reason. To see this consider equation (6) in the baseline model. This equation implies that if liquidity traders sell \( u_F \) shares of security \( F \) then the price of security \( D \) will drop by \( C_{D1} \times u_F \), other things equal. Thus, in the baseline model as well, a drop in the price of security \( F \) (due to for instance to a large liquidity sell order in this security) will propagate to security \( D \). The propagation however is due to cross-asset learning: Dealers in security \( D \) observe the drop in price for security \( F \), revise downward their estimate of \( \delta_F \), and therefore the payoff of security \( D \).

These channels for price spillovers are well-known in the literature and co-exist in reality. More important for our purposes, cross-market arbitrageurs do not generate liquidity spillovers. Indeed, in the absence of dealers, the illiquidity of security \( j \) is:

\[ \frac{\partial p_j}{\partial u_j} = \frac{1 + d_j^2 + \sigma^2_{\eta_j}}{\lambda \gamma_A}, \] (28)

Thus, the illiquidity of security \( D \) does not directly affect the illiquidity of security \( F \) and vice versa. Hence, there is no illiquidity multiplier and no illiquidity spillovers. In particular, a change in parameters (\( d \) or \( \sigma^2_{\eta_j} \)) that does not directly affect the illiquidity of security \( F \) has no effect on the illiquidity of security \( D \).

4.2.2 Cross market arbitrageurs and the illiquidity multiplier

We now consider the case in which arbitrageurs coexist with dealers and all dealers are price-watchers (\( \mu_F = \mu_D = 1 \)). As in the baseline model, we focus on the linear rational expectations

\[ p_D = p_F = \frac{2}{\lambda \gamma_A} (u_D + u_F) \] if \( u_D + u_F < 0 \), the price of each asset is smaller than its expected payoff (zero) in order to induce cross-market arbitrageurs to hold the shares of assets \( D \) and \( F \) sold by liquidity traders (cross-market arbitrageurs' portfolio is risky even if the two assets are perfect substitutes). The prices of both assets however must be equal as otherwise a riskless arbitrage opportunity would exist.

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\[35\] If both assets are perfect substitutes, that is, \( d_j = d = 1 \) and \( \sigma^2_{\eta_j} = 0 \), the prices of both assets are identical and equal to \( p_D = p_F = \frac{2}{\lambda \gamma_A} (u_D + u_F) \). If \( u_D + u_F < 0 \), the price of each asset is smaller than its expected payoff (zero) in order to induce cross-market arbitrageurs to hold the shares of assets \( D \) and \( F \) sold by liquidity traders (cross-market arbitrageurs' portfolio is risky even if the two assets are perfect substitutes). The prices of both assets however must be equal as otherwise a riskless arbitrage opportunity would exist.
equilibria of the model. The following proposition extends Proposition 1 when \( \lambda > 0 \).

**Proposition 3.** When \( \mu_D = \mu_F = 1 \) and \( \lambda > 0 \), there always exists one or three non-fully revealing linear rational expectations equilibrium. An equilibrium is such that

\[
p_{j1}^A = R_j^A \delta_j + R_j^A B_{j1} u_j + E_j^A \delta_{-j} + E_j^A B_{-j1} u_{-j},
\]

(29)

where the coefficients, \( E_j^A \), and \( R_j^A \) can be expressed as functions of \( B_{j1} \) and \( B_{-j1} \), which are defined as in Proposition 1 (that is solve (7) and (8)).

Thus, the set of parameters for which multiple equilibria obtain is exactly the same as in the baseline model since the coefficients \( B_{j1} \)s solve the same system of equations and determine all other coefficients in equilibrium. The reason is that cross-market arbitrageurs do not change the self-reinforcing mechanism between prices and liquidity obtained in the baseline version of the model since this mechanism derives from cross-asset learning and not cross-market hedging. As a result, the sets of parameters for which there is a unique equilibrium or multiple equilibria are identical to those in the baseline case.

Using equation (29), we obtain that the illiquidity of market \( j \) is given by

\[
\frac{\partial p_{j1}^A}{\partial u_j} \equiv R_j^A B_{j1}.
\]

(30)

For \( \lambda = 0 \), \( R_j^A = 1 \) and the illiquidity of each security is as in the baseline model. For \( \lambda > 0 \), \( R_j^A < 1 \) and the illiquidity of each security is smaller than in the baseline model. This is intuitive since cross-market arbitrageurs also act as liquidity providers. Thus, as \( \lambda \) increases, the illiquidity of both assets becomes smaller relative to the baseline case, as shown by Figure 6.

![Figure 6: Illiquidity in market D (panel (a)) and F (panel (b)) for \( \lambda \in \{0, 0.01, \ldots, 2\} \). Other parameter values: \( \sigma_{u_j} = 2 \), \( \sigma_{\eta_D} = d_D = 1 \), \( \gamma_j = \gamma_A = 1 \), \( \sigma_{\eta_F} = d_F = 0 \).](image)

Finally, in Figure 7, we plot the resulting covariance for the illiquidity of securities \( D \) and \( F \) against \( \lambda \) assuming that \( \sigma_{\eta_D} \) has a uniform distribution over \([0.01, 2.5]\). When only arbitrageurs are
present in the market, this covariance is zero since a change in $\sigma_{\eta_D}$ has no effect on the illiquidity of security $F$ (see equation (28) in Section 4.2.1). When arbitrageurs and dealers co-exist, this covariance is in general strictly positive but, as expected, it decreases as the amount of capital dedicated to arbitrage increases.

Figure 7: Comovement in illiquidity. The figure displays the covariance between the illiquidity of security $F$ and the illiquidity of security $D$ as a function of $\lambda$ in the market with only arbitrageurs (panel (a)), and in the market with arbitrageurs and pricewatchers (panel (b)). Parameters’ values are as follows: $\gamma_D = \gamma_F = \gamma_A = 1/2$, $\sigma_{u_j} = 1$, $d_D = .9$, $\sigma_{\eta_D} \in \{0.01, 0.02, \ldots, 2.5\}$, $\lambda \in \{0.01, 0.02, \ldots, 2\}$, and $\sigma_{\eta_F} = d_F = 0$.

Overall cross-market trading does not qualitatively change the conclusions obtained in the baseline model. However, cross-market arbitrageurs dampen co-movements in liquidity coming from dealers’ cross-asset learning.

5 Conclusions

Our paper is the first to show how liquidity spillovers arise in a multi-asset framework when dealers in different assets learn from each others’ prices. Specifically, we show that there is a self-reinforcing relationship between the liquidity of two assets with correlated payoffs. The reason is that when one asset becomes less liquid, its price becomes less informative for dealers specialized in the other asset. Thus, these dealers face more uncertainty and they require larger price concessions to absorb liquidity traders’ order imbalances. Thus, an increase in illiquidity propagates from the first asset to the second. In turn, by the same mechanism, the increase in illiquidity of the second asset feeds back on the first, amplifying the initial shock.

This propagation and amplification mechanism has several implications. First, it can lead to liquidity melt-downs or liquidity melt-ups for all assets, following a small shock affecting the liquidity of only one asset. Second, it predicts that co-movements in liquidity should increase with the extent to which dealers in one asset class can make their prices contingent on the prices
of other asset classes and should decrease with the capital allocated to cross-market arbitrage. Third, it implies that a change in market structure or regulations affecting the liquidity of one asset class can affect another asset class, even if it is not directly affected by the structural change. Our paper suggests several interesting directions for future research. For instance, it would be interesting to measure empirically the strength of liquidity spillovers across asset classes and to check that liquidity is more volatile when spillovers are stronger, as implied by our model. Another interesting issue is how the number of assets affects the amplification mechanism described in our paper. In particular, this analysis could help to identify whether there are assets that play a more important role in liquidity spillovers, maybe because their prices are followed by more dealers or maybe because their payoff structure makes them more correlated with other assets.
Appendix A: No cross-market arbitrageurs.

Proof of Lemma 1.
If $\mu_j = 0$ then all dealers in security $j$ only observe factor $\delta_j$ when they choose their demand function. As dealers have a CARA utility function, it is immediate that their demand function in this case is

$$x^j_I(\delta_j) = \gamma_j \frac{E[v_j|\delta_j] - p_j}{\text{Var}[v_j|\delta_j]} = \gamma_j \left( \frac{\delta_j - p_j}{\sigma_{\eta_j}^2 + d_j^2} \right).$$  \hspace{1cm} (A.1)

Imposing market clearing, we deduce that the equilibrium price is given by:

$$p_j = \delta_j + \left( \frac{\sigma_{\eta_j}^2 + d_j^2}{\gamma_j} \right) u_j = \delta_j + B_{j0} u_j,$$

with

$$B_{j0} = \frac{\sigma_{\eta_j}^2 + d_j^2}{\gamma_j}.$$

□

Proof of Propositions 1 and 2.
Proposition 1 is a special case of Proposition 2, which considers the more general case in which $\mu_j$ can take any value. Thus, we directly prove Proposition 2. The proof is standard. That is, (i) we conjecture that the equilibrium price is such that $p^*_j = R_j \delta_j + B_j u_j + E_j \delta_{-j} + C_j u_{-j}$ with $R_j = 1, C_j = E_j B_{-j},$ and $B_j > 0$; (ii) we compute dealers’ demand functions given this conjecture; (iii) we deduce using these demand functions and the clearing condition that our conjecture is correct if $B_j$ is given by equation (20). When $\mu_D = \mu_F = 1$, this equation yields the system of equations (7) and (8) in Proposition 1. Last, we show that the system of equations implied by equation (20) always has a solution, which proves the existence of a linear rational expectations equilibrium.

**Step 1.** Let $\hat{\omega}_j = B_j u_j + E_j \omega_{-j}$ and $\omega_j = \delta_j + B_j u_j$. As $C_j = E_j B_{-j}$ (to be shown), the conjectured equilibrium price in market $j$ can be written $p^*_j = \delta_j + \hat{\omega}_j = \omega_j + E_j \omega_{-j}$. As pricewatchers know the prices in both markets, they can deduce signals $\omega_j$ and $\omega_{-j}$ from these prices. For pricewatchers in security $j$, $\omega_j$ is not informative since they already know $\delta_j$. In contrast $\omega_{-j}$ is informative about $\delta_{-j}$. Thus, $\{\delta_j, \omega_{-j}\}$ is a sufficient statistic for pricewatchers’ information set $\{\delta_j, p^*_j, p_j^*\}$. Inattentive dealers in security $j$ just observe the price in their market. As they know $\delta_j$, they extract the signal $\hat{\omega}_j$ from $p^*_j$. Hence, $\{\delta_j, \hat{\omega}_j\}$ is a sufficient statistic for inattentive dealers’ information set $\{\delta_j, p^*_j\}$.

**Step 2.** Equilibrium in market $j$.
Pricewatchers’ demand function. A pricewatcher’s demand function in market $j$, $x^W_j(\delta_j, p_j, p_{-j})$, maximizes

$$E \left[ - \exp \left\{ - \frac{(v_j - p_j) x^W_j}{\gamma_j} \right\} | \delta_j, p_j, p_{-j} \right].$$
We deduce that
\[ x_j^W(\delta_j, p_j, p_{-j}) = \gamma_j \left( \frac{E[v_j|\delta_j, p_{-j}, p_j] - p_j}{\text{Var}[v_j|\delta_j, p_{-j}]} \right) \]
\[ = a_j^W(\delta_j, p_{-j}, p_j) - p_j, \]
with \( a_j^W = \gamma_j \text{Var}[v_j|\delta_j, p_{-j}]^{-1} \). As \( \{\delta_j, \omega_{-j}\} \) is a sufficient statistic for \( \{\delta_j, p_{-j}, p_j\} \), we deduce (using well-known properties of normal random variables) that
\[ E[v_j|\delta_j, p_{-j}, p_j] = E[v_j|\delta_j, \omega_{-j}] \]
\[ = \delta_j + \frac{d_j}{(1 + B_j^2 \sigma_{u,j}^2)} \omega_{-j}, \]
(A.3)
and
\[ a_j^W = \frac{\gamma_j}{\text{Var}[v_j|\delta_j, \omega_{-j}]} \]
\[ = \gamma_j \left( \frac{1}{d_j^2 B_j^2 \sigma_{u,j}^2 + \sigma_j^2 (1 + B_j^2 \sigma_{u,j}^2)} \right) \]
\[ = \frac{\gamma_j}{\text{Var}[v_j|\delta_j](1 - \rho_j)}, \]
(A.4)
where \( \rho_j \equiv d_j^2/((\sigma_{u,j}^2 + d_j^2)(1 + B_j^2 \sigma_{u,j}^2)) \). Thus,
\[ x_j^W(\delta_j, \omega_{-j}) = a_j^W(\delta_j - p_j) + b_j^W \omega_{-j}, \]
where
\[ b_j^W = \frac{\gamma_j \text{Cov}[v_j, \omega_{-j}]}{\text{Var}[v_j|\delta_j, \omega_{-j}]} \]
\[ = d_j a_j^W \left( \frac{1}{1 + B_j^2 \sigma_{u,j}^2} \right). \]
(A.5)

**Inattentive Dealers.** An inattentive dealer demand function in market \( j \), \( x_j^I(\delta_j, p_j) \), maximizes:
\[ E \left[ - \exp \left\{ - \left( (v_j - p_j) x_j^I \right) / \gamma_j \right\} | \delta_j, p_j \right]. \]
We deduce that
\[ x_j^I(\delta_j, p_j) = \gamma_j \left( \frac{E[v_j|\delta_j, p_j] - p_j}{\text{Var}[v_j|\delta_j, p_{-j}]} \right) \]
\[ = a_j^I(\delta_j, p_j) - p_j, \]
(A.6)
with \( a_j^I = \gamma_j \text{Var}[v_j|\delta_j, p_{-j}]^{-1} \). As \( \{\delta_j, \tilde{\omega}_j\} \) is a sufficient statistic for \( \{\delta_j, p_j\} \), we deduce (using well-known properties of normal random variables) that
\[ E[v_j|\delta_j, p_j] = E[v_j|\delta_j, \tilde{\omega}_j] \]
\[ = \delta_j + \frac{d_j E_j}{E_j^2 (1 + B_j^2 \sigma_{u,j}^2) + B_j^2 \sigma_{u,j}^2} \tilde{\omega}_j, \]
(A.7)
and

\[ a_j^I = \frac{\gamma_j}{\text{Var}[v_j | \delta_j, \hat{\omega}_j]} \]

\[ \gamma_j \sigma^2(u_{-j}) + B_j^2 \sigma^2_{u_j} \]

\[ E_j^2(1 + B^{-2}_j \sigma^2_{u_{-j}}) + B_j^2 \sigma^2_{u_j} \].

(A.8)

Thus,

\[ x_j^I(\delta_j, \hat{\omega}_j) = a_j^I(\delta_j - p_j) + b_j^I \hat{\omega}_j, \]

where

\[ b_j^I = \frac{\gamma_j}{\text{Var}[v_j | \delta_j, \hat{\omega}_j]} \frac{\text{Cov}[v_j, \hat{\omega}_j]}{\text{Var}[\hat{\omega}_j]} \]

\[ = \frac{a_j^I \sigma^2_{u_j}(1 + B^{-2}_j \sigma^2_{u_{-j}}) + B_j^2 \sigma^2_{u_j}}{E_j^2(1 + B^{-2}_j \sigma^2_{u_{-j}}) + B_j^2 \sigma^2_{u_j}}. \]

(A.9)

**Clearing price in market** \( j \). The clearing condition in market \( j \in \{D, F\} \) imposes

\[ \mu_j x_j^W(\delta_j, p_j, p_{-j}) + (1 - \mu_j)x_j^I(\delta_j, p_j) + u_j = 0. \]

Let \( a_j = \mu_j a_j^W + (1 - \mu_j)a_j^I \). Using equations (A.2) and (A.6), we solve for the clearing price and we obtain

\[ p_j^* = \delta_j + \left( \frac{\mu b_j^W + (1 - \mu_j)b_j^I E_j}{a_j} \right) \omega_j + \left( \frac{(1 - \mu_j)b_j^I B_j + 1}{a_j} \right) u_j, \]

(A.10)

Remember that we are searching for an equilibrium such that \( p_j^* = R_j \delta_j + B_j u_j + E_j \delta_{-j} + C_j u_{-j} \).

We deduce from equation (A.10) that in equilibrium, we must have \( R_j = 1 \),

\[ B_j = \left( \frac{(1 - \mu_j)b_j^I B_j + 1}{a_j} \right), \quad E_j = \left( \frac{\mu b_j^W + (1 - \mu_j)b_j^I E_j}{a_j} \right), \] and \( C_j = E_j B_{-j} \).

Thus

\[ B_j = \frac{1}{a_j - (1 - \mu_j)b_j^I}, \quad \text{for} \ j \in \{D, F\}, \]

(A.11a)

\[ E_j = \mu_j B_j b_j^W, \quad \text{for} \ j \in \{D, F\}. \]

(A.11b)

We show below that coefficients \( a_j^W, a_j^I, b_j^W \) and \( b_j^I \) only depend on \( B_D \) and \( B_F \). Hence, coefficients \( E_j \) and \( C_j \) can be expressed only in terms of \( \{B_D, B_F\} \), which means that the equilibrium is fully characterized once \( B_D \) and \( B_F \) are known.

It is immediate from (A.4) that \( a_j^W \) only depends on \( B_{-j} \). Moreover, substituting (A.4) in (A.5) and rearranging we obtain

\[ b_j^W = \frac{d_j \gamma_j}{d_j^2 B^{-2}_j \sigma^2_{u_{-j}} + \sigma^2_{u_j} (1 + B^{-2}_j \sigma^2_{u_{-j}})}. \]

(A.12)
Using (A.11b) and (A.12), we can rewrite (A.9) as

\[ b_j^l = a_j^l \frac{d_j^2 \mu_j \gamma_j (d_j^2 B_{-j} \sigma_{u,-j}^2 + \sigma_{\eta_j}^2 (1 + B_{-j}^2 \sigma_{u,-j}^2))}{B_j (\mu_j^2 d_j^2 \gamma_j^2 (1 + B_{-j}^2 \sigma_{u,-j}^2) + \sigma_{u_j}^2 (d_j^2 B_{-j}^2 \sigma_{u,-j}^2 + \sigma_{\eta_j}^2 (1 + B_{-j}^2 \sigma_{u,-j}^2))^2)}. \]  

(A.13)

Similarly, using (A.11b) and (A.12), we can rewrite (A.8) as

\[ a_j^f = \frac{\gamma_j (\mu_j^2 \rho_j + \sigma_{u_j}^2 (d_j^2 + \sigma_{\eta_j}^2) (1 - \rho_j)^2)}{(d_j^2 + \sigma_{\eta_j}^2) (1 - \rho_j) (\mu_j^2 \gamma_j \rho_j + \sigma_{u_j}^2 (d_j^2 + \sigma_{\eta_j}^2) (1 - \rho_j))}. \]

(A.14)

Inserting (A.14) in (A.13) yields after some algebra

\[ b_j^f = \frac{d_j^2 \mu_j}{\gamma_j B_j (\mu_j^2 d_j^2 \gamma_j^2 \sigma_{u,-j}^2 + \sigma_{\eta_j}^2 (\sigma_{\eta_j}^2 + d_j^2)(\sigma_{\eta_j}^2 + d_j^2)(1 + B_{-j}^2 \sigma_{u,-j}^2) + d_j^2 B_{-j}^2 \sigma_{u,-j}^2)). \]

(A.15)

Hence, as claimed previously, coefficients \( a_j^W, a_j^f b_j^W \) and \( b_j^f \) only depend on \( B_j \) and \( B_{-j} \). We can now replace (A.4), (A.14) and (A.15) in (A.11a) and, after some tedious algebra, we obtain

\[ B_j = f_j(B_{-j}; \mu_j, \gamma_j, \sigma_{\eta_j}, d_j, \sigma_{u_j}), \]

(A.16)

where

\[ f_j(B_{-j}; \mu_j, \gamma_j, \sigma_{\eta_j}, d_j, \sigma_{u_j}, \sigma_{u,-j}) = \frac{B_{j0}(1 - \rho_j)(\mu_j^2 \gamma_j \rho_j + (\sigma_{\eta_j}^2 + d_j^2)\sigma_{u_j}^2 (1 - \rho_j))}{\rho_j \mu_j^2 \gamma_j^2 + \sigma_{u_j}^2 (\sigma_{\eta_j}^2 + d_j^2) (1 - \rho_j) (1 - \rho_j (1 - \mu_j))}, \]

(A.17)

with \( \rho_j = d_j^2/((\sigma_{\eta_j}^2 + d_j^2)(1 + B_{-j}^2 \sigma_{u,-j}^2)) \) and \( B_{j0} = (\sigma_{\eta_j}^2 + d_j^2)/\gamma_j \). Last, as \( \text{Var}[v_j \delta_j] = \sigma_{\eta_j}^2 + d_j^2 \), we obtain that

\[ B_j = B_{j0}(1 - \rho_j) \times \frac{\gamma_j^2 \mu_j \rho_j + \sigma_{u_j}^2 \text{Var}[v_j \delta_j] (1 - \rho_j)}{\gamma_j^2 \mu_j \rho_j + \sigma_{u_j}^2 \text{Var}[v_j \delta_j] (1 - \rho_j) (1 - \rho_j (1 - \mu_j))}, \]

(A.18)

as claimed in Proposition 2. When \( \mu_D = \mu_F = 1 \), equation (A.18) written for \( j = D \) and \( j = F \) yields equations (7) and (8) in Proposition 1.

Step 3. Existence and number of non-fully revealing equilibria when \( \mu_D = \mu_F = 1 \).

Let \( f_{j1}(B_{-j}; \mu_j, \gamma_j, \sigma_{\eta_j}, d_j, \sigma_{u,-j}) \equiv f_j(B_{-j}; 1, \gamma_j, \sigma_{\eta_j}, d_j, \sigma_{u_j}, \sigma_{u,-j}) \) where \( f_j(\cdot) \) is defined in equation (A.17), where we drop \( \sigma_{u,-j} \) as an argument of \( f_{j1}(\cdot) \) because \( \sigma_{u,-j} \) does not affect \( f_j(\cdot) \) when \( \mu_j = 1 \) (see equation (A.17)). When \( \mu_D = \mu_F = 1 \), we deduce from equation (A.16) that a non-fully revealing rational expectations equilibrium exists if and only if the following system of equations has a strictly positive solution

\[ B_{D1} = f_{D1}(B_{F1}; \gamma_D, \sigma_{\eta_D}, d_D, \sigma_{u_F}) = \frac{\sigma_{\eta_D}^2}{\gamma_D} + \frac{d_D^2 B_{F1}^2 \sigma_{u_F}^2}{\gamma_D (1 + B_{F1}^2 \sigma_{u_F}^2)}, \]

(A.19)

\[ B_{F1} = f_{F1}(B_{D1}; \gamma_F, \sigma_{\eta_F}, d_F, \sigma_{u_D}) = \frac{\sigma_{\eta_F}^2}{\gamma_F} + \frac{d_F^2 B_{D1}^2 \sigma_{u_D}^2}{\gamma_F (1 + B_{D1}^2 \sigma_{u_D}^2)}. \]

(A.20)
Let’s define the function $h_1(B_{D1}) \equiv f_{D1}(f_{F1}(B_{D1}))$:

$$h_1(B_{D1}) = \frac{\sigma_{\eta D}^2 \gamma_F^2 (1 + B_{D1}^2 \sigma_{uD}^2) + \sigma_{uF}^2 (\sigma_{\eta D}^2 + d_D^2)(\sigma_{\eta F}^2 (1 + B_{D1}^2 \sigma_{uD}^2) + d_F^2 B_{D1}^2 \sigma_{uD}^2)^2}{\gamma_D (\gamma_F^2 (1 + B_{D1}^2 \sigma_{uD}^2)^2 + (\sigma_{\eta F}^2 (1 + B_{D1}^2 \sigma_{uD}^2) + d_F^2 B_{D1}^2 \sigma_{uD}^2)^2 \sigma_{uF}^2)}. \quad (A.21)$$

Equation (A.19) implies that in equilibrium, $B_{D1}$ solves $h_1(B_{D1}) = B_{D1}$. Thus, the number of non-fully revealing equilibria is equal to the number of times $h_1(B_{D1})$ cuts the 45-degree line for values of $B_{D1} > 0$. It is immediate that

$$h_1(0) = \frac{\sigma_{\eta D}^2 (\gamma_F^2 + \sigma_{\eta F}^4 \sigma_{uF}^2) + d_D^2 \sigma_{\eta F}^4 \sigma_{uF}^2}{\gamma_D (\gamma_F^2 + \sigma_{\eta F}^4 \sigma_{uF}^2)} > 0,$$

and it can be verified that $h_1(B_{D0}) < B_{D0} \equiv (\sigma_{\eta D}^2 + d_D^2)/\gamma_D$. Thus, there always exists a noisy rational expectations equilibrium with $B_{D1}$ in the interval

$$\left(\frac{\sigma_{\eta D}^2 (\gamma_F^2 + \sigma_{\eta F}^4 \sigma_{uF}^2) + d_D^2 \sigma_{\eta F}^4 \sigma_{uF}^2}{\gamma_D (\gamma_F^2 + \sigma_{\eta F}^4 \sigma_{uF}^2)}, B_{D0}\right). \quad (A.22)$$

We now show that the second derivative of $h_1(\cdot)$ changes sign only once. Coupled with the fact that $h_1'(B_{D1}) > 0$, this property of $h_1(\cdot)$ implies that the curve representing this function cuts the 45-degree line in either one or three points. Thus, there can be either 1 or 3 equilibria. Differentiating $h_1(B_{D1})$ twice, we obtain:

$$h''_1(B_{D1}) \propto (\gamma_F^2 \sigma_{uD} \sigma_{uF} d_D d_F)^2 \times (-12 B_{D1}^2 \sigma_{uD}^8 (d_F^2 + \sigma_{\eta F}^2)(\gamma_F^2 (d_F^2 + \sigma_{\eta F}^2)^2 \sigma_{uF}^2)$$

$$- 12 B_{D1}^2 \sigma_{uD}^4 (\gamma_F^2 (3d_F^2 + 8 \sigma_{\eta F}^2) + (d_F^2 + \sigma_{\eta F}^2)^2 (5d_F^2 + 8 \sigma_{\eta F}^2) \sigma_{uF}^2)$$

$$- 12 B_{D1}^4 \sigma_{uD}^4 (\gamma_F^2 (3d_F^2 + 2 \sigma_{\eta F}^2)^2 \sigma_{uF}^2 - \gamma_F^2 (d_F^2 - 2 \sigma_{\eta F}^2))$$

$$+ 12 B_{D1}^2 \sigma_{uD}^2 (\gamma_F^2 - \sigma_{\eta F}^2 \sigma_{uF}^2) + 4 \sigma_{\eta F}^4 (\gamma_F^2 + \sigma_{\eta F}^4 \sigma_{uF}^2)). \quad (A.23)$$

Note that (A.23) is a 8-degree polynomial that has negative coefficients at its terms of power 8 and 6, while the coefficient of the power-0 term is positive. We now show that if the coefficient of the term of power 2 is negative, the same sign is inherited by the coefficient of the term of power 4. To see this, suppose that

$$\gamma_F^2 < \sigma_{\eta F}^4 \sigma_{uF}^2.$$

As a consequence we have

$$\gamma_F^2 d_F^2 < \sigma_{\eta F}^4 \sigma_{uF}^2 d_F^2 < 2 \gamma_F^2 \sigma_{\eta F}^2 + \sigma_{\eta F}^2 (d_F^2 + \sigma_{\eta F}^2)(3d_F^2 + 2 \sigma_{\eta F}^2) \sigma_{uF}^2.$$

Thus, in this case we have that the coefficients of the polynomial (A.23) change sign only once, implying that by Descartes’ rule of signs the polynomial has a unique real root. If on the other hand $\gamma_F^2 \geq \sigma_{\eta F}^2 \sigma_{uF}^2$ once again the coefficients of the polynomial change sign only once, and again we claim that $h''(B_{D1}) = 0$ has only one root. 

□
Step 4. Existence of a non fully revealing equilibrium when $\mu_j < 1$.

Let define the function

$$
\Psi(B_D) \equiv f_{D}(f_{F}(B_D; \mu_F; \gamma_F; \sigma_{\eta_F}, d_F, \sigma_{u_F}, \sigma_{u_D}); \mu_D, \gamma_D, \sigma_{\eta_D}, d_D, \sigma_{u_D}, \sigma_{u_F}) - B_D.
$$

We deduce from equation (A.16) that a non-fully revealing equilibrium exists if and only if the ratio of two polynomials and that $\Psi(\cdot)$ is continuous. Moreover:

$$
\Psi(0) > 0, \text{ and } \lim_{B_D \to \infty} \Psi(B_D) = -\infty,
$$

for either $\sigma_{\eta_D} > 0$ or $\sigma_{\eta_F} > 0$. Thus, as $\Psi(\cdot)$ is continuous, there exists at least one value of $B_D$ such that $\Psi(B_D) = 0$.

Proof of Lemma 2

Remember that the equilibrium price in market $j$ can be written $p_j^* = \omega_j + E_j \omega_{-j}$ (see Step 1 in the proof of Propositions Proposition 1 and 2). As $\omega_{-j} = p_{-j}^* - E_{-j} \omega_j$, we can also write the equilibrium price in market $j$ as

$$
p_j^* = \omega_j + E_j (p_{-j}^* - E_{-j} \omega_j) = (1 - E_j E_{-j}) \omega_j + E_j p_{-j}^*.
$$

Proof of Corollary 1

The result follows immediately from equation (13).

Proof of Corollary 2

The result follows immediately from equation (12).

Proof of Corollary 3

From Step 4 of the proof of Proposition 1 we deduce that there is a unique rational expectations equilibrium iff the equation $\Psi(B_{D1}) = 0$ as a unique solution, where $\Psi(B_{D1}) \equiv h_1(B_{D1}) - B_{D1}$ and $h_1(\cdot)$ is defined in equation (A.21). Thus, if $\Psi'(\cdot)$ is everywhere negative, the equilibrium is unique. Tedious calculations show that $\Psi'(\cdot)$ is proportional to:

$$
\Psi'(B_{D1}) \propto -\gamma_D \gamma_F^2 (1 + B_{D1}^2 \sigma_{u_D}^2)^2 - 4 B_{D1} (\gamma_D B_{D1} - \sigma_{\eta_D}^2) \sigma_{u_D}^2 (\gamma_F^2 (1 + B_{D1}^2 \sigma_{u_D}^2) + (\sigma_{\eta_F}^2 + d_F^2) \sigma_{u_F}^2 \times \\
(\sigma_{\eta_F}^2 (1 + B_{D1}^2 \sigma_{u_D}^2) + d_F^2 B_{D1}^2 \sigma_{u_D}^2)) - \sigma_{u_F}^2 (\sigma_{\eta_F}^2 (1 + B_{D1}^2 \sigma_{u_D}^2) + d_F^2 B_{D1}^2 \sigma_{u_D}^2) \times
\\
(\gamma_D (\sigma_{\eta_F}^2 (1 + B_{D1}^2 \sigma_{u_D}^2) + d_F^2 B_{D1}^2 \sigma_{u_D}^2) - 4 \sigma_{u_D}^2 d_D B_{D1} (\sigma_{\eta_F}^2 + d_F^2)).
$$

(A.25)

We deduce that $\Psi'(B_{D1}) < 0$ if

$$
\gamma_D (\sigma_{\eta_F}^2 (1 + B_{D1}^2 \sigma_{u_D}^2) + d_F^2 B_{D1}^2 \sigma_{u_D}^2) - 4 \sigma_{u_D}^2 d_D B_{D1} (\sigma_{\eta_F}^2 + d_F^2) > 0,
$$

(A.26)
since the two first terms in (A.25) are negative and the last term is negative as well if Condition (A.26) is satisfied. This is the case if

\[ B_{D1}^2 \sigma_{u_D}^2 \gamma_D (\sigma_{\eta_F}^2 + d_F^2) - 4B_{D1} \sigma_{u_D}^2 d_F^2 (\sigma_{\eta_F}^2 + d_F^2) > 0, \]

or

\[ B_{D1} > \frac{4d_F^2}{\gamma_D}. \]

From equation (A.22) we know that at equilibrium

\[ B_{D1} > \frac{\sigma_{\eta_D}^2 (\gamma_F^2 + \sigma_{\eta_F}^4 \sigma_{u_F}^2) + d_F^2 \sigma_{\eta_F}^4 \sigma_{u_F}^2}{\gamma_D (\gamma_F^2 + \sigma_{\eta_F}^4 \sigma_{u_F}^2)}. \]

Therefore, the equilibrium is unique if

\[ \frac{\sigma_{\eta_D}^2 (\gamma_F^2 + \sigma_{\eta_F}^4 \sigma_{u_F}^2) + d_F^2 \sigma_{\eta_F}^4 \sigma_{u_F}^2}{\gamma_D (\gamma_F^2 + \sigma_{\eta_F}^4 \sigma_{u_F}^2)} > \frac{4d_F^2}{\gamma_D}. \]

That is,

\[ \frac{\sigma_{\eta_D}^2}{d_F^2} > \frac{(4\gamma_F^2 + 3\sigma_{\eta_F}^4 \sigma_{u_F}^2)}{\gamma_F^2 + \sigma_{\eta_F}^4 \sigma_{u_F}^2}. \]

By symmetry, the condition:

\[ \frac{\sigma_{\eta_F}^2}{d_F^2} > \frac{(4\gamma_D^2 + 3\sigma_{\eta_D}^4 \sigma_{u_D}^2)}{\gamma_D^2 + \sigma_{\eta_D}^4 \sigma_{u_D}^2}, \]

is also sufficient. Combining these two conditions yield Corollary 3.

\[ \square \]

Proof of Corollary 4

The result follows immediately from the definition of \( f_{j1}(\cdot) \) in Proposition 1.

\[ \square \]

Proof of Corollary 5

Step 1. The total effect of a change in the liquidity fundamental, \( \chi_j \), on the illiquidity of security \( j \) is given by

\[ \frac{dB_{j1}}{d\chi_j} = \frac{\partial f_{j1}}{\partial \chi_j} + \frac{\partial f_{j1}}{\partial B_{-j1}} \cdot \frac{dB_{-j1}}{d\chi_j}. \]

As

\[ \frac{dB_{-j1}}{d\chi_j} = \frac{\partial f_{-j1}}{\partial B_{j1}} \cdot \frac{dB_{j1}}{d\chi_j}, \]

we deduce that

\[ \frac{dB_{j1}}{d\chi_j} = \kappa \frac{\partial f_{j1}}{\partial \chi_j}, \]

\[ \frac{dB_{-j1}}{d\chi_j} = \kappa \times \frac{\partial f_{-j1}}{\partial B_{j1}} \frac{\partial f_{j1}}{\partial \chi_j}, \]
with \( \kappa = 1 - ((\partial f_{-j1}/\partial B_{j1})(\partial f_{j1}/\partial B_{-j1})) \). If \( d_j = 0 \), we have \( \partial f_{j1}/\partial B_{-j1} = 0 \) and \( \kappa = 1 \).

**Step 2:** We now consider the case in which \( d_j > 0 \). Remember that by definition

\[
\eta(B_{D1}) \equiv f_{D1}(f_{F1}(B_{D1}; \gamma_F, \sigma_{DF}, d_F, \sigma_{DF}); \gamma_D, \sigma_{DF}, d_D, \sigma_{DF}).
\]

Note that

\[
\frac{\partial \eta}{\partial B_{D1}} = \frac{\partial f_{D1}}{\partial B_{F1}} \frac{\partial f_{F1}}{\partial B_{D1}}.
\]

Hence, we have \( \kappa > 1 \) if and only if \( \eta'(B_{D1}) < 1 \) at an equilibrium value for \( B_{D1} \). A necessary condition for this is \( d_j > 0 \), \( j \in \{D, F\} \), as otherwise \( \partial f_{j1}/\partial B_{-j1} = 0 \). As explained in the proof of Proposition 1 (see Step 3), the equilibrium values for \( B_{D1} \) are obtained at the points where \( \eta(B_{D1}) \) crosses the 45-degree line and there are either one such point or three.

Let’s consider first a case in which there are three equilibria and let’s call the equilibrium values of \( B_{D1} \) in this case, \( B_{D1}^L \), \( B_{D1}^M \), and \( B_{D1}^H \) with \( B_{D1}^L < B_{D1}^M < B_{D1}^H \). We have shown in Step 3 of the proof of Proposition 1 that \( \eta''(B_{D1}) \) changes sign only once. Denote by \( B_{D1} \) the value at which \( \eta''(B_{D1}) \) changes sign. Using the expression for \( \eta''(B_{D1}) \) in Step 3 of the proof of Propositions 1 it is easily seen that \( \eta'(B_{D1}) > 0 \) for \( B_{D1} \) small. Thus, \( \eta''(B_{D1}) > 0 \) for \( B_{D1} < B_{D1} \). Moreover, \( B_{D1}^M < B_{D1} \). Hence, at \( B_{D1}^L \) and \( B_{D1}^M \), \( \eta(\cdot) \) is convex. As \( \eta(0) > 0 \), it means that \( \eta(\cdot) \) cuts the 45-degree line for the first time from above and the second time from below. That is, \( 0 < \eta'(B_{D1}) < 1 \) and \( \eta'(B_{D1}) > 1 \). Moreover, we have \( B_{D1} < B_{D1} \) since \( \eta(\cdot) \) passes above the 45-degree line at \( B_{D1}^M \) and crosses it again at \( B_{D1}^H \). Hence, the curvature of \( \eta(\cdot) \) must change in the interval \( (B_{D1}^M, B_{D1}^H) \). At \( B_{D1}^H \), the function \( \eta(\cdot) \) cuts the 45-degree line from above since it cuts its from below at \( B_{D1}^M \). Hence, \( 0 < \eta'(B_{D1}) < 1 \).

Now consider the case in which there is only one equilibrium. The analysis is identical except that \( \eta(\cdot) \) cuts the 45-degree line only at one point \( B_{D1}^* \). At this point \( \eta(\cdot) \) is either concave or convex, but since \( \eta(0) > 0 \), it must be that \( \eta(\cdot) \) cuts the 45-degree line from above. That is, \( 0 < \eta'(B_{D1}^*) < 1 \).

To sum up, if the equilibrium is unique then it must be such that \( \kappa \geq 1 \). If instead there are three equilibria, only the two extreme equilibria (those for which \( B_{D1} = B_{D1}^L \) or \( B_{D1} = B_{D1}^H \)) are such that \( \kappa > 1 \). \( \Box \)

**Proof of Corollary 6**

From equation (17):

\[
\kappa(B_{D1}, B_{F1}) \equiv \left(1 - \frac{\partial f_{F1}}{\partial B_{D1}} \frac{\partial f_{D1}}{\partial B_{F1}}\right)^{-1}. \quad (A.27)
\]

When \( \sigma_{uj} = \sigma_u, \gamma_j = \gamma, \sigma_{u-j} = \sigma_u, \) and \( d_j = d \), the markets for the two assets are symmetric. In this case, we have:

\[
f_{j1}(B_{-j1}; \gamma, \sigma_u, d, \sigma_u) = \frac{\sigma_u^2}{\gamma} + \frac{\partial^2 B_{-j1}^2 \sigma_u^2}{\gamma(1 + B_{-j1}^2 \sigma_u^2)}.
\]

\(^{36}\)Otherwise, \( \eta''(B_{D1}) \) would change sign before cutting the 45-degree line for the second time, which would imply that it never cuts the 45-degree line more than once. This is impossible in a case with three equilibria.
Therefore, we have:

\[ \frac{\partial f_{j1}}{\partial B_{-j1}} = \left( \frac{2d^2\sigma_u^2}{\gamma} \right) \left( \frac{B_{-j1}}{1 + B_{-j1}^2\sigma_u^2} \right)^2. \]

Straightforward calculations show that \( \frac{\partial f_{j1}}{\partial B_{-j1}} \) is concave in \( B_{-j1} \) and reaches its maximum for \( B_{-j1}^\text{max} = (\sqrt{3}\sigma_u)^{-1}. \) We deduce from equation \( (A.27) \) that if there is an equilibrium such \( B_D = B_F = (\sqrt{3}\sigma_u)^{-1} \) then this equilibrium is such that \( \kappa \) reaches its largest possible equilibrium value, denoted \( \kappa^\text{max} \), when the exogenous parameters are fixed and such that \( \sigma_{\eta_j} = \sigma_{\eta}, \gamma_j = \gamma, \sigma_{u_{-j}} = \sigma_u, \) and \( d_j = d. \)

The equilibrium \( B_D = B_F = (\sqrt{3}\sigma_u)^{-1} \) is obtained iff \( \hat{\gamma} = \frac{\sqrt{3}\sigma_u(d^2 + 4\sigma_u^2)}{4}. \) This can be easily seen by checking that equations \((7)\) and \((8)\) are satisfied for \( B_D = B_F = (\sqrt{3}\sigma_u)^{-1}\) when \( \gamma = \hat{\gamma}. \)

Furthermore, using equation \( (A.27) \), we deduce a closed-form solution for \( \kappa^\text{max} \), namely:

\[
\begin{align*}
\kappa^\text{max} &= \left( 1 - \left( \frac{\partial f_{j1}}{\partial B} \right)^2 \right)^{-1}
\bigg|_{B = B_{-j1}^\text{max} \ \gamma = \hat{\gamma}} \\
&= \left( 1 - \frac{27d^4\sigma_u^2}{64\hat{\gamma}^2} \right)^{-1} \\
&= \left( 1 - \frac{9d^4}{4(4\sigma_u^2 + d^2)^2} \right)^{-1}.
\end{align*}
\]

(A.28)

(A.29)

Let \( d^\circ \) be the value of \( d \) such that the denominator of \( \kappa^\text{max} \) is zero. Straightforward calculations show that \( d^\circ = 2\sqrt{2}\sigma_u \) and \( \kappa^\text{max} \) is larger than one only if \( d < d^\circ \) since the denominator of \( \kappa^\text{max} \) increases with \( d. \) Clearly as \( d \to (d^\circ)^- \), then \( \kappa^\text{max} \to \infty \) and for \( d > d^\circ, \) \( \kappa^\text{max} \) is negative.

**Proof of Corollary 7**

Observe that a change in \( B_{-j} \) only affects the illiquidity of security \( j \) through its effect on \( \rho_j^2. \) As \( \rho_j^2 \) declines in \( B_{-j}, \) we deduce that liquidity spillovers from security \( j \) to security \( -j \) are positive if and only if \( \frac{\partial B_j}{\partial \rho_j} < 0. \) Now we show that \( \mu_j \geq \bar{\mu}_j \) is a sufficient condition for this to be the case. Observe that \( B_j = B_{j0}(1 - \rho_j)G(\mu_j, \rho_j) \) with

\[ G(\mu_j, \rho_j) = \frac{\gamma_j^2(\mu_j + \rho_j)\sigma_{u_j}^2 \text{Var}[\eta_j|\delta_j](1 - \rho_j)}{2\gamma_j^2\mu_j^2\rho_j + \sigma_{u_j}^2 \text{Var}[\eta_j|\delta_j](1 - \rho_j)(1 - \rho_j(1 - \mu_j))}. \]

(A.30)

Therefore, we have:

\[
\frac{\partial B_j}{\partial \rho_j} = -B_{j0}G(\mu_j, \rho_j) + B_{j0}(1 - \rho_j)\frac{\partial G}{\partial \rho_j}.
\]

(A.31)

Now observe that

\[
\frac{\partial G(\mu_D, \rho_D)}{\partial \rho_D} = \frac{(\sigma_{\eta_D}^2 + \mu_D^2)(1 - \mu_D)(1 - \rho_D)^3\sigma_{u_D}^2}{(\gamma_D^2\mu_D^2(1 + \rho_D) + (\sigma_{\eta_D}^2 + \mu_D^2)(1 - \rho_D)^3\sigma_{u_D}^2(1 - \mu_D))^2} > 0.
\]

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Inserting this expression and the expression for \(G(\mu_D, \rho_D^2)\) in equation (A.31), we obtain after some algebra

\[
\frac{\partial B_D}{\partial \rho_D} = -\frac{\text{Var}[v_D|\delta_D]\mu_D}{\gamma_D(\gamma_D^2 \mu_D^2 \rho_D + \sigma_u^2 \text{Var}[v_D|\delta_D](1-\rho_D)(1-\rho_D(1-\mu_D)))^2} \times \\
(\gamma^4 \mu_D^2 \rho_D^2 + \sigma_u^2 \text{Var}[v_D|\delta_D](1-\rho_D)(\text{Var}[v_D|\delta_D](1-\rho_D)\sigma_u^2 - \gamma_D^2(1-\mu_D - \rho_D(1+\mu_D)))).
\]

As \(\rho_D < 1\), we deduce that the sign of \(\frac{\partial B_D}{\partial \rho_D}\) is the opposite of the sign of

\[
\mu_D - \left(\frac{R_D - 1}{R_D}\right) \left(\frac{1-\rho_D}{1+\rho_D}\right),
\]

which is positive if \(\mu_D \geq \mu_D\). We deduce that \(\frac{\partial f}{\partial B_F} > 0\) if \(\mu_D > \mu_D\). A similar reasoning shows that \(\frac{\partial g}{\partial B_D} > 0\) if \(\mu_F > \mu_F\). \(\Box\)
B Appendix: Dealers and cross-market arbitrageurs

B.1 Benchmark: No dealers

We first analyze the case in which there are no dealers. Prices contain no information in this case since arbitrageurs have no information. Hence, an arbitrageur’s optimal portfolio solves

$$\max_{\mathbf{x}^A} E \left[ -\exp \left\{ -\left( \frac{1}{\gamma_A} \right) (\mathbf{x}^A)^T (\mathbf{v} - \mathbf{p}) \right\} \right],$$

where superscript $T$ is used to designate the transpose of a vector. The FOC of this problem yields

$$\mathbf{x}^A = \gamma_A \text{Var}[\mathbf{v}]^{-1} (E[\mathbf{v}] - \mathbf{p}),$$

where $\mathbf{x}^A$ is the $2 \times 1$ vector of arbitrageurs’ positions in each asset and

$$\text{Var}[\mathbf{v}] = \begin{pmatrix} 1 + d_D^2 + \sigma_{\eta_D}^2 & d_D + d_F \\ d_D + d_F & 1 + d_F^2 + \sigma_{\eta_F}^2 \end{pmatrix}.$$ 

As $E[\mathbf{v}|\Omega^A] = E[\mathbf{v}] = 0$, we obtain

$$\mathbf{x}^A = \gamma_A \frac{\lambda \gamma_A}{(1 - d_D d_F)^2 + \sigma_{\eta_D}^2 (1 + d_D^2) + \sigma_{\eta_F}^2 (1 + d_F^2) + \sigma_{\eta_D}^2 \sigma_{\eta_F}^2} \times \left( \begin{array}{c} -(1 + d_D^2 + \sigma_{\eta_D}^2) p_D + (d_D + d_F) p_F \\ -(1 + d_F^2 + \sigma_{\eta_F}^2) p_D + (d_D + d_F) p_F \end{array} \right), \quad (B.1)$$

that is,

$$x_D^A(p_D, p_F) = \gamma_A \frac{\lambda \gamma_A}{(1 - d_D d_F)^2 + \sigma_{\eta_D}^2 (1 + d_D^2) + \sigma_{\eta_F}^2 (1 + d_F^2) + \sigma_{\eta_D}^2 \sigma_{\eta_F}^2} \times \left( -(1 + d_D^2 + \sigma_{\eta_D}^2) p_D + (d_D + d_F) p_F \right), \quad (B.1)$$

and

$$x_F^A(p_D, p_F) = \gamma_A \frac{\lambda \gamma_A}{(1 - d_D d_F)^2 + \sigma_{\eta_D}^2 (1 + d_D^2) + \sigma_{\eta_F}^2 (1 + d_F^2) + \sigma_{\eta_D}^2 \sigma_{\eta_F}^2} \times \left( -(1 + d_F^2 + \sigma_{\eta_F}^2) p_F + (d_D + d_F) p_D \right). \quad (B.2)$$

Using the clearing conditions in each market, we deduce that the equilibrium prices of securities $D$ and $F$ are:

$$p_D = \frac{1 + d_D^2 + \sigma_{\eta_D}^2}{\lambda \gamma_A} u_D + \frac{d_D + d_F}{\lambda \gamma_A} u_F \quad (B.3a)$$

$$p_F = \frac{1 + d_F^2 + \sigma_{\eta_F}^2}{\lambda \gamma_A} u_F + \frac{d_D + d_F}{\lambda \gamma_A} u_D. \quad (B.3b)$$

□
B.2 Proof of Proposition [3]

Step 1. In linear rational expectations equilibria, the demand functions of the dealers are linear functions of the prices of each asset and their private signals. Denote by \( x^W_j (\delta_j, p_j, p_{-j}) = a^W_j \delta_j - \phi^W_j (p_j, p_{-j}) \) the demand function of asset \( j \) by dealers in this asset where \( \phi^W_j (\cdot) \) is linear in both prices. Similarly the demand functions for each asset of the cross-market arbitrageurs are linear in the prices of each asset. Denote by \( x^A_j = \phi^A_j (p_j, p_{-j}) \) these demand functions where \( \phi^A_j (\cdot) \) is linear in both prices.

Let \( B_D = 1/a^W_D \) and \( B_F = 1/a^W_F \). The clearing conditions in each market imply

\[
\begin{align*}
    a^W_D \omega_D + \lambda \varphi^A_D (p_D, p_F) &= \varphi^W_D (p_D, p_F) \quad \text{(B.4a)} \\
    a^W_F \omega_F + \lambda \varphi^A_F (p_D, p_F) &= \varphi^W_F (p_D, p_F). \quad \text{(B.4b)}
\end{align*}
\]

As \( \varphi^W_j (\cdot) \) and \( \varphi^A_j (\cdot) \) are linear functions, we immediately deduce from the system of equations (B.4a) and (B.4b) that observing the prices of securities \( D \) and \( F \) is observationally equivalent to \( \{ \omega_D, \omega_F \} \). Signal \( \omega_j \) does not contain new information for dealers in security \( j \) but signal \( \omega_{-j} \) does. Hence \( \Omega^W_j = \{ \delta_j, \mathcal{P}_D^W \} = \{ \delta_j, \omega_{-j} \} \). As cross-market arbitrageurs only observe prices, we also deduce that \( \Omega^A = \{ p_D, p_F \} = \{ \omega_D, \omega_F \} \).

Step 2. We now use these remarks to show that any linear rational expectations equilibrium has the following form

\[
\begin{align*}
    p_D &= R^A_D \omega_D + E^A_D \omega_F, \quad \text{(B.6a)} \\
    p_F &= R^A_F \omega_F + E^A_F \omega_D, \quad \text{(B.6b)}
\end{align*}
\]

where \( A^H_j \) and \( R^H_j \) are equilibrium coefficients and \( \omega_D \) and \( \omega_F \) are as defined in equations (B.5a) and (B.5b). The problem of pricewatchers is exactly as in the baseline case. We deduce that the optimal demand of pricewatchers in security \( j \) is

\[
\begin{align*}
    x^W_j &= \gamma_j \frac{E[v_j|\Omega^W_j] - p_j}{\text{Var}[v_j|\Omega^W_j]} \\
    &= \frac{1}{B_j} (\delta_j - p_j) + b^W_j \omega_{-j}, \quad \text{(B.7)}
\end{align*}
\]

where

\[
B_j = \frac{\text{Var}[v_j|\{\delta_j, \omega_{-j}\}]}{\gamma_j}, \quad b^W_j = \frac{1}{B_j} \frac{\text{Cov}[v_j, \omega_{-j}]}{\text{Var}[\omega_{-j}]}.
\]

An arbitrageurs’ optimal portfolio, \( x^A = (x^A_D, x^A_F)^T \) solves

\[
\max_{x^A} E \left[ -\exp \left\{ -(1/\gamma_A)(x^A)^T (v - p) \right\} | \Omega^H \right],
\]
which yields

\[ \mathbf{x}^A = \mathbf{\Gamma}^A (E[v|\Omega^A] - \mathbf{p}), \]

where \( \mathbf{\Gamma}^A = \gamma_A \text{Var}[v|\Omega^A]^{-1} \). As all random variables have a normal distribution, standard properties of conditional moments for these variables yield

\[ E[v|\Omega^A] = \begin{pmatrix} H_{DD} & H_{DF} \\ H_{FD} & H_{FF} \end{pmatrix} \begin{pmatrix} \omega_D \\ \omega_F \end{pmatrix}, \]

with

\[ \begin{pmatrix} H_{DD} & H_{DF} \\ H_{FD} & H_{FF} \end{pmatrix} = \begin{pmatrix} \text{Cov}[v_D, \omega_D]/\text{Var}[\omega_D] & \text{Cov}[v_D, \omega_F]/\text{Var}[\omega_F] \\ \text{Cov}[v_F, \omega_D]/\text{Var}[\omega_D] & \text{Cov}[v_F, \omega_F]/\text{Var}[\omega_F] \end{pmatrix} \]

\[ = \begin{pmatrix} 1/\text{Var}[\omega_D] & d_D/\text{Var}[\omega_F] \\ d_F/\text{Var}[\omega_D] & 1/\text{Var}[\omega_F] \end{pmatrix}. \]

Furthermore

\[ \mathbf{\Gamma}^A = \gamma_A \text{Var}[v|\Omega^A]^{-1} \]

\[ = \gamma_A \left( \text{Var}[v] - \text{Cov}[v, \{\omega_D, \omega_F\}] \right)^{-1} \text{Var}[\{\omega_D, \omega_F\}] \]

\[ = \begin{pmatrix} a^A_{D1} & a^A_{D2} \\ a^A_{F1} & a^A_{F2} \end{pmatrix}. \]

We deduce that an arbitrageur’s optimal positions in securities \( D \) and \( F \) are

\[ x^A_D = a^A_{D1} (E[v_D|\Omega^A] - p_D) + a^A_{D2} (E[v_F|\Omega^A] - p_F) \quad \text{(B.9a)} \]

\[ x^A_F = a^A_{F1} (E[v_F|\Omega^A] - p_F) + a^A_{F2} (E[v_D|\Omega^A] - p_D) \quad \text{(B.9b)} \]

where

\[ a^A_{D1} = \frac{\gamma_A}{\Delta} \left( \frac{B_F^2 \sigma_{u_F}^2}{1 + B_F^2} + \frac{d_F^2 B_D^2 \sigma_{u_D}^2}{1 + B_D^2 \sigma_{u_D}^2} + \sigma_{\eta_F}^2 \right) \quad \text{(B.10a)} \]

\[ a^A_{D2} = a^A_{F2} = -\frac{\gamma_A}{\Delta} \left( \frac{d_D B_F^2 \sigma_{u_F}^2}{1 + B_F^2 \sigma_{u_F}^2} + \frac{d_F B_D^2 \sigma_{u_D}^2}{1 + B_D^2 \sigma_{u_D}^2} \right) \quad \text{(B.10b)} \]

\[ a^A_{F1} = \frac{\gamma_A}{\Delta} \left( \frac{B_F^2 \sigma_{u_D}^2}{1 + B_F^2} + \frac{d_D^2 B_F^2 \sigma_{u_F}^2}{1 + B_F^2 \sigma_{u_F}^2} + \sigma_{\eta_D}^2 \right). \quad \text{(B.10c)} \]

and

\[ \Delta = \left( \frac{B_F^2 \sigma_{u_F}^2}{1 + B_F^2} + \frac{d_F^2 B_D^2 \sigma_{u_D}^2}{1 + B_D^2 \sigma_{u_D}^2} + \sigma_{\eta_F}^2 \right) \left( \frac{B_D^2 \sigma_{u_D}^2}{1 + B_D^2} + \frac{d_D^2 B_F^2 \sigma_{u_F}^2}{1 + B_F^2 \sigma_{u_F}^2} + \sigma_{\eta_D}^2 \right) \]

\[ - \left( \frac{d_D B_F^2 \sigma_{u_F}^2}{1 + B_F^2 \sigma_{u_F}^2} + \frac{d_F B_D^2 \sigma_{u_D}^2}{1 + B_D^2 \sigma_{u_D}^2} \right)^2. \quad \text{(B.11)} \]

The clearing conditions impose:

\[ x^W_D + \lambda x^A_D + u_D = 0 \quad \text{(B.12a)} \]

\[ x^W_F + \lambda x^A_F + u_F = 0, \quad \text{(B.12b)} \]
Substituting (B.7), (B.9a), and (B.9b) in (B.12a) and (B.12b) and collecting terms yields the following system of equations

\[
\begin{align*}
\Phi_{D3}p_D &= \Phi_{D1}\omega_D + \Phi_{D2}\omega_F - \Phi_{D4}p_F, \\
\Phi_{F3}p_F &= \Phi_{F1}\omega_F + \Phi_{F2}\omega_D - \Phi_{F4}p_D,
\end{align*}
\] (B.13a, B.13b)

where

\[
\begin{align*}
\Phi_{D1} &= a_D^W + \lambda(a_{D1}^A H_{DD} + a_{D2}^A H_{DF}), & \Phi_{F1} &= a_F^W + \lambda(a_{F1}^A H_{FF} + a_{F2}^A H_{DF}) \\
\Phi_{D2} &= b_D^W + \lambda(a_{D1}^A H_{DF} + a_{D2}^A H_{FF}), & \Phi_{F2} &= b_F^W + \lambda(a_{F1}^A H_{FD} + a_{F2}^A H_{DD}) \\
\Phi_{D3} &= a_D^W + \lambda a_{D1}^A, & \Phi_{D4} &= \lambda a_{D2}^A, \quad \Phi_{F3} = a_F^W + \lambda a_{F1}^A, \quad \Phi_{F4} = \lambda a_{F2}^A.
\end{align*}
\]

Solving for the equilibrium prices yields

\[
\begin{align*}
p_D &= \frac{\Phi_{D1}\Phi_{F3} - \Phi_{D4}\Phi_{F2}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}} \omega_D + \frac{\Phi_{D2}\Phi_{F3} - \Phi_{D4}\Phi_{F1}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}} \omega_F, \\
p_F &= \frac{\Phi_{D3}\Phi_{F1} - \Phi_{D2}\Phi_{F4}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}} \omega_F + \frac{\Phi_{D3}\Phi_{F2} - \Phi_{D1}\Phi_{F4}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}} \omega_D.
\end{align*}
\] (B.14a, B.14b)

Our conjecture on the form of the linear rational expectations is correct if and only if:

\[
\begin{align*}
R_D^A &= \frac{\Phi_{D1}\Phi_{F3} - \Phi_{D4}\Phi_{F2}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}}, & E_D &= \frac{\Phi_{D2}\Phi_{F3} - \Phi_{D4}\Phi_{F1}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}}, \\
R_F^A &= \frac{\Phi_{D3}\Phi_{F1} - \Phi_{D2}\Phi_{F4}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}}, & E_F &= \frac{\Phi_{D3}\Phi_{F2} - \Phi_{D1}\Phi_{F4}}{\Phi_{D3}\Phi_{F3} - \Phi_{D4}\Phi_{F4}},
\end{align*}
\] (B.15)

which proves our claim.

**Step 3.** Observe that the coefficients \(R_j^A\) and \(E_j^A\) given in equation (B.15) can be written ultimately only in terms of \(B_D\) and \(B_F\) since coefficients \(a_{j1}^A, a_{j2}^A,\) and \(b_j^W\) are known once \(B_D\) and \(B_F\) are known. Moreover, using equation (B.8), we have:

\[
B_j = \frac{\text{Var}[v_j | \{\delta_j, \omega_j\}]}{\gamma_j}
\]

That is:

\[
\begin{align*}
B_{D1} &= \frac{\sigma_{\eta_D}^2}{\gamma_D} + \frac{d_D^2 B_{F1}^2 \sigma_{u_F}^2}{\gamma_D (1 + B_{F1}^2 \sigma_{u_F}^2)}, \\
B_{F1} &= \frac{\sigma_{\eta_F}^2}{\gamma_F} + \frac{d_F^2 B_{D1}^2 \sigma_{u_D}^2}{\gamma_F (1 + B_{D1}^2 \sigma_{u_D}^2)}.
\end{align*}
\] (B.16a, B.16b)

Thus, \(B_{D1}\) and \(B_{F1}\) solve the same system of equations as in the baseline case. Thus, as in the baseline case, there is either one of three non fully revealing linear rational expectations equilibria when \(\sigma_{\eta_D}^2 > 0\) or \(\sigma_{\eta_F}^2 > 0\).
References


