Consider an agent with a forward position of an illiquid asset (e.g. a commodity) that has to be closed before delivery. Suppose that the liquidity of the asset increases as the delivery date approaches. Assume further that the agent has two possibilities for hedging the risk inherent in the forward position: first, he can enter customized forward contracts; second, he can acquire standardized and liquidly traded forward contracts. We assume that purchasing customized forwards perfectly eliminates the risk, but entails high liquidity costs charged by the counterparty. The standardized forwards can be acquired at considerably lower costs, but do not perfectly match the agent’s risk and hence entail basis risk. By means of stochastic control we show how to obtain an optimal trade-off between liquidity costs and basis risk. To this end we reduce the hedging problem to a family of stopping problems. In two case studies we consider simple liquidity dynamics for which optimal hedging strategies can be calculated explicitly.

Keywords: illiquidity, basis risk, optimal liquidation, hedging, singular stochastic control, optimal stopping.

Introduction

The revenues from gas power plants strongly depend on natural gas prices. Gas power plant operators usually protect themselves against rising prices by buying natural gas on
forward markets. However, in Europe there are many different natural gas market areas, differing considerably in liquidity. To buy or sell natural gas on the forward market with delivery in one of the less liquid market areas often calls for considerable liquidity premia. An operator of a gas power plant located in one of the less liquid areas, e.g. one of the German market areas, may therefore consider to cross hedge the price risk by buying natural gas in a more liquid neighboring area, e.g. at the Dutch Title Transfer Facility (TTF). As soon as the German market becomes liquid, typically shortly before delivery, the operator can sell the Dutch gas again and buy the German gas he needs. Prices in neighboring market areas are highly correlated, with the price spread being essentially determined by the transportation costs from one area to the other. The risk inherent in a German short position, therefore, can be strongly reduced by cross hedging it with a Dutch long position. Future price spreads, however, are uncertain and hence any geographic cross hedge entails basis risk. The operator thus has to decide between two risk reducing strategies: a perfect hedge in the domestic market area implying higher liquidity costs, and a cross hedge with natural gas from a more liquid neighboring area entailing basis risk.

Indeed, it happens frequently that companies can choose between different ways of transferring their risk to markets. Many companies have forward risk positions and it is a crucial task of their risk managers to assess the risk exposure and to develop strategies to keep it in acceptable bounds. In this paper we consider situations where risk managers can decide between the following two possibilities for reducing the risk of an illiquid asset position:

- By paying a liquidity surcharge, the risk can be hedged perfectly. Since there is no liquid market for the particular risk, the manager has to find a counterparty that is willing to enter an over-the-counter (OTC) contract, e.g. a forward contract. For entering the contract the counterparty will ask for a premium that usually strongly depends on liquidity. The more alternative agents and hence potential risk takers, the smaller the liquidity costs.

- The risk can be cross hedged by using standardized products traded on a liquid market, e.g. standardized forwards or exchange-traded futures. The standardized products can be acquired with less costs, but they do not provide a perfect risk protection. Therefore, the risk managers face the trade-off between a perfect hedge with higher liquidity costs and a cross hedge implying basis risk.

Whether agents make a perfect or a cross hedge, depends on how they expect the liquidity costs to evolve until the risk is due. Typically trading of forward contracts becomes more active as the contract’s delivery period approaches. More intensive trading increases liquidity and hence reduces execution costs. This is also consistent with the Samuelson effect according to which the volatility of a forward price increases as maturity approaches (see [10]).

On trading platforms usually only contracts with consecutive delivery periods (weeks, months, quarters respectively years) are fixed. Only front periods (e.g. the following three months) are traded actively. When a new month starts, trading of a new month
(e.g. the third front month) forward contract sets in. Trading of a particular forward, therefore, can become active suddenly.

The paper aims at describing the optimal trade-off between the two hedging possibilities. Our goal is to provide simple and explicit decision rules that can guide practitioners how to hedge their risk. To this end we need to make some simplifying assumptions, e.g. in how we model liquidity. We interpret liquidity costs as half of the bid-ask spread. In other words, the liquidity costs for selling respectively buying one asset share are equal to the absolute difference of the realized price to the mid-market price. We assume that the bid-ask spread is an exogenously given stochastic process. We do not assume that it depends on the order size. Allowing in addition for a volume-dependent price impact would make it difficult to obtain explicit hedging strategies; one would have to fall back on numerical methods. Furthermore, we believe that this assumption is a reasonable approximation of reality in many cases, in particular for the application of our model to the gas forward market in Europe outlined above.

We will first set up a general model with stochastic liquidity costs and develop a method for calculating hedging strategies. We then apply the method to two stylized case studies in which we derive optimal hedges explicitly.

In the first case study (see Section 3) we assume that trading becomes suddenly active at a random time $\tilde{\tau}$: before $\tilde{\tau}$ liquidity costs are constant equal to a high level $K^+$, after $\tilde{\tau}$ equal to a lower level $K^-$. We model the liquidity jump as the first jump time of a Poisson process. In the second case study (see Section 4) we assume that liquidity costs are deterministic non-increasing functions. We study the influence of the speed with which transaction costs decay. We distinguish increasing and decreasing speed; in other words, costs that are concave respectively convex over time.

The optimal position paths are determined by the time decay of the risk and the liquidity costs. For example, suppose that the speed of the risk decay is constant and the speed of the cost decay is increasing. If it is not optimal to close the position of the illiquid asset immediately (at 0), then it must be optimal to close it at the latest possible time (at $T$). Whether the position is closed at time 0 or at time $T$ depends on the initial ratio of liquidity costs and risk. In this particular case we obtain a simple decision rule of the form:

- If the liquidity costs of the cross hedging instrument, measured in terms of its bid-ask spread, are smaller than a given threshold $L$, then the risk position should be cross hedged.

- If the bid-ask spread exceeds $L$, then no cross hedge should be performed. Whether the illiquid position is closed or not, depends on the ratio between the liquidity cost savings and the risk when keeping the position open.

The decision rule can be illustrated by a simple decision tree (see e.g. Figure 1 below).

Our approach for obtaining optimal hedging strategies is as follows: We formulate the agent’s aim of finding optimal positions as a singular control problem. The well-known
connection between singular control and optimal stopping, see e.g. [8],[5],[6] and the references therein, can also be established within our model: We show that the problem of finding optimal position strategies is equivalent to a family of stopping problems. Within the two case studies we have explicit solutions of the associated stopping problems and we can thus derive optimal hedging strategies in closed-form.

The trade-off of basis risk against liquidity costs when closing an illiquid forward position seems not have been studied yet. There is a branch of literature concerned with how to close large asset positions (without taking into account cross hedging possibilities). Frequently, see e.g. [4], [1], [11], [9] and [3], trades are supposed to entail a temporary price impact: the price at which a trade is executed depends on the size of the order. For simplicity we do not assume a price impact in our model; instead, we assume that execution costs are proportional to the size.

On the other hand, there are many papers dealing with how to optimally cross hedge risk if no perfect hedging instruments are at hand. For an overview on quadratic approaches we refer to [12], for utility based approaches see e.g. [2] and the references therein.

The importance to consider liquidity costs in hedging is stressed in [13]: the authors show that liquidity costs of futures and futures options on wheat (traded at Kansas City board of trade) differ considerably.

The remainder of the paper is organized as follows. In Section 1 we set up a general market model; we assume that the liquidity of the asset which has to be closed by the investor is a non-increasing stochastic process. We describe the trade-off between the two hedging possibilities by a cost functional which results in a singular stochastic control problem. In Section 2 we analyze to which extent this problem is equivalent to a family of stopping problems. In particular we describe an iterative method for determining optimal hedging strategies. We then employ the method within stylized examples in order to derive explicit decision rules: in Section 3 we assume that liquidity jumps randomly from a high to a lower level; in Section 4 we suppose that the liquidity is deterministic, and that the cost decay over time is either convex or concave. The proofs of the results of Sections 3 and 4 are presented in Appendix A and B respectively.

1. A model with stochastic liquidity

Consider an agent aiming at closing a short forward position of an illiquid asset (e.g. German natural gas in the example of the introduction). We suppose that there is an OTC forward market, where one can buy and sell the asset. We further assume that there exists a more standardized and liquidly traded asset that is highly correlated with the asset to be hedged. We will refer to the illiquid asset as the primary asset, and to the liquid one as the proxy of the illiquid asset.

Let $x_0 < 0$ be the initial short position of the primary asset. We assume that the agent has to close the position latest at time $T > 0$. The agent has the choice between buying the illiquid asset on the forward market before time $T$ or on the spot market.
at time $T$. The spot price may also involve some transaction costs. We denote by $K_t$ the costs arising from a closure of one unit at $t \in [0,T]$. We assume that $K$ is a non-negative adapted stochastic process on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]})$, where $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness. Moreover we suppose that the paths of $K$ are càdlàg on $[0,T]$ (i.e. they are right-continuous and possess left-hand limits).

The proxy is assumed to be liquidly traded. Nevertheless, any acquisition of the proxy will entail transaction costs. For simplicity we assume that liquidity of the proxy is constant over the period $[0,T]$, and we denote by $L \in \mathbb{R}_+$ half the bid ask spread. Besides we assume that the agent must not have any proxy position at time $T$ (e.g. because of a physical settlement)\(^1\).

By a position process of the primary asset we mean any $\mathbb{F}$-adapted process $X : [0,T] \times \Omega \to \mathbb{R}$ that is càdlàg and satisfies $X_T = 0$. Analogously, a proxy position $Y : [0,T] \times \Omega \to \mathbb{R}$ is a càdlàg $\mathbb{F}$-adapted process satisfying $Y_T = 0$. We suppose that the initial cross hedge position, the proxy position, is zero. We define $X_{0-} = x_0$ and $Y_{0-} = 0$. Any pair $(X,Y)$ satisfying the properties above will be referred to as a position strategy. The set of all position strategies will be denoted by $\mathcal{D}(x_0)$.

The overall execution costs entailed by a strategy $(X,Y)$ are given by

$$C(X,Y) = \int_{[0,T]} K_s |dX_s| + L \int_{[0,T]} |dY_s|,$$

where $|dX_s|$ denotes the integral with respect to the total variation of the path $X$ over the whole interval $[0,T]$. Note that the integral includes the boundary of the interval $[0,T]$, which means that

$$\int_{[0,T]} K_s |dX_s| = K_0 |X_0 - X_{0-}| + \int_{(0,T)} K_s |dX_s| + K_T |X_T - X_{T-}|.$$

Throughout we will assume that the forward price processes of the primary asset and the proxy are martingales. Thus, the returns have zero expectation and the liquidation is not affected by any directional views about the price processes. A model with a non-zero drift can result in profits from trading even if no initial position is to be closed; this makes it difficult to differentiate between optimal liquidation and optimal investment. The additional analysis of optimal investment is not the focus of this article (cf. the motivating discussion about gas forward markets in Europe above). We denote by $\sigma_1$ (respectively $\sigma_2$) the standard deviation of the primary asset (respectively proxy) forward price increment over $[0,T]$. Let $\rho$ be the correlation between the primary asset and the proxy and denote the covariance matrix by

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}.$$

\(^1\)Cf. the motivating example about different gas market areas in Europe outlined in the introduction: the forward position in Dutch gas has to be closed before the gas is physically delivered. Our model does not treat the case where forwards are cash settled, i.e. where the proxy position does not need to be closed before delivery. It can, however, be easily extended to this case.
Throughout we assume that $\rho \geq 0$.

We measure the risk associated to a portfolio consisting of $X_t$ primary and $Y_t$ proxy shares with the help of the quadratic form

$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \Sigma \begin{pmatrix} x \\ y \end{pmatrix} = \sigma_1^2 x^2 + 2\rho \sigma_1 \sigma_2 xy + \sigma_2^2 y^2.$$  

Note that if the price increments are Gaussian, then $f(x, y)$ is the variance of the portfolio value.

Let $g : [0, \infty) \to [0, \infty)$ be a continuous, non-decreasing function with $g(0) = 0$, which is continuously differentiable on $(0, \infty)$. Moreover assume that $x \mapsto g(f(x, y))$ is convex for all $y \geq 0$. The risk associated to a position strategy $(X, Y)$ will be defined by

$$E \int_0^T g(f(X_s, Y_s))ds,$$

where $E$ is the expectation operator. We are in particular interested in the case $g(x) = \lambda \sqrt{x}$, $\lambda \geq 0$ (cf. the case studies of Sections 3.3 and Section 4). This choice corresponds to a first order approximation of the position’s value at risk.

Observe that for a given primary position $X$, the hedge position that minimizes the portfolio’s risk is given by

$$Y_t = -\rho \frac{\sigma_1}{\sigma_2} X_t.$$

We define $h = \rho \frac{\sigma_1}{\sigma_2}$ and remark that $h$ is frequently referred to as the minimum variance hedge ratio (see e.g. [7], Chapter 3).

We suppose that the agent aims at minimizing the sum of the expected execution costs and the portfolio’s risk. For $(X, Y) \in \mathcal{D}(x_0)$ we define the objective functional

$$J(X, Y) = E \left[ \int_{[0,T]} K_s|dX_s| + \int_{[0,T]} L|dY_s| + \int_0^T g(f(X_s, Y_s))ds \right].$$

Notice that the first two integrals in (1) include a possible jump at time 0.

The value function is defined by

$$v(x_0) = \inf_{(X,Y)\in \mathcal{D}(x_0)} J(X,Y).$$

As usual, we say that a position strategy is optimal if it attains the infimum in (2).

Optimal strategies in general are not absolutely continuous with respect to the Lebesgue measure. We are, therefore, dealing with a singular stochastic control problem. There is a well-established link between singular control and optimal stopping (see e.g. [8],[3],[6] and the references therein). In the following section we establish this link within our model and reduce the problem of finding an optimal position strategy to a family of stopping problems.

We close this section by observing that if forward prices are martingales, then a strategy minimizing executions costs also minimizes the agent’s expected overall costs for closing the short position.
Remark 1.1. We assume that the price of the primary asset \((P_t)_{t \in [0,T]}\) is a continuous martingale. The agent’s overall costs from following a position process \(X\) amount to \(\int_{[0,T]} P_s dX_s + \int_{[0,T]} K_s |dX_s|\). Integrating by parts and using that \(X_T = 0\) yields

\[
\int_{[0,T]} P_s dX_s = -P_0 x_0 - \int_0^T X_s - dP_s.
\]

Under suitable integrability assumptions on \(X\) the process \(t \mapsto \int_0^t X_s - dP_s\) is a martingale starting in 0. This implies

\[
E \left[ \int_{[0,T]} P_s dX_s + \int_{[0,T]} K_s |dX_s| \right] = -P_0 x_0 + E \left[ \int_{[0,T]} K_s |dX_s| \right],
\]

and, hence, the expected overall costs are the difference of the expected execution costs and the initial book value \(P_0 x_0\). Similar considerations hold true for the proxy position, which shows that (under suitable assumptions) minimizing the expected overall costs is equivalent to minimizing just the expected execution costs.

2. Optimal positions via optimal stopping

In this section we show that the problem of finding optimal position processes is equivalent to a family of stopping problems. To this end we first show that any optimal primary position path is non-decreasing, and any optimal proxy position is at first non-decreasing and then non-increasing (Section 2.1). This allows us then to encode the optimal position strategy by stopping times (Sections 2.2 and 2.3). Finally, we present a method to successively determine the optimal position paths (Section 2.4).

2.1. Optimal position paths are (piecewise) monotone

We next show that the optimal position process \(X\) of the primary must be non-decreasing. On the other hand, the optimal position for the proxy is only non-decreasing until the “optimal” hedging position \(-hX\) is reached; afterwards the process is non-increasing: at all times it is given by \(-hX\). In particular, this confirms the intuition that the optimal hedge is at most the minimum variance hedge ratio.

We first require the following notation. For \(z \in \mathbb{R}\) denote by \(A(z)\) the set of stochastic processes on \([0,T]\) that are adapted, càdlàg, non-decreasing and satisfy \(Z_t \geq z\) for all \(t \in [0,T]\). We set \(Z_{0-} = z\). If \(z \leq 0\), then we denote by \(A_0(z)\) the subset of processes \(Z \in A(z)\) with \(Z_T = 0\).

Proposition 2.1. We have

\[
v(x_0) = \inf \{ J(X, I \land -hX) | (X, I) \in A_0(x_0) \times A(0) \}.
\]
Proof. Let \((X, Y)\) be an arbitrary pair of position paths in \(D(x_0)\). Let \(\hat{X}\) be the smallest adapted, càdlàg and non-decreasing process dominating \(X\) (i.e. \(\hat{X}_t = \sup_{0 \leq s \leq t} X_s\)) and define

\[
\hat{Y}_t = Y_t \land -h \hat{X}_t.
\]

The execution costs entailed by \((\hat{X}, \hat{Y})\) are smaller than the costs of \((X, Y)\). Moreover, we have that \(f(\hat{X}_t, \hat{Y}_t) \leq f(X_t, Y_t)\), which implies that also the risk associated with \((\hat{X}, \hat{Y})\) is smaller than the risk associated with \((X, Y)\). Therefore, \(J(\hat{X}, \hat{Y}) \leq J(X, Y)\).

Next define a new cross hedge via \(\tilde{Y}_t = \sup_{0 \leq s \leq t} \tilde{Y}_s \land -h \hat{X}_t\). Then the execution costs entailed by \((\hat{X}, \tilde{Y})\) are smaller than or equal to the costs of \((\hat{X}, \hat{Y})\). Moreover, we have \(f(\hat{X}_t, \tilde{Y}_t) \leq f(\hat{X}_t, \hat{Y}_t)\). This shows that there exists a strategy of the form \(Y_t = I_t \land -h \hat{X}_t\) with \(I \in A(0)\) such that \(Y\) is at least as good as \(\hat{Y}\), i.e. \(J(\hat{X}, Y) \leq J(\hat{X}, \hat{Y})\).

Instead of determining optimal position paths in both assets simultaneously, we first look at the problem of finding reciprocal optimal positions. This will give us some qualitative insights in the shape of position paths that will allow us to determine an optimal solution, at least in some cases.

We first explain how one can encode positions paths via family of stopping times. For any \(Z \in A(z)\) we define the associated family of stopping times

\[
\tau(y) = \tau^Z(y) = \inf\{t \geq 0 | Z_t > y\}, \quad \text{for all } y \in [z, \infty)
\]

with the convention \(\inf\emptyset = +\infty\). Observe that the mapping \([z, \infty) \ni y \mapsto \tau(y)\) is right-continuous and non-decreasing, and hence càdlàg. The process \(Z\) can be recovered from \((\tau(y))\), namely

\[
Z_t = \inf\{y \geq z | \tau(y) \geq t\}. \quad (3)
\]

Indeed, any family of stopping times \((\tau(y))_{y \geq z}\), such that \([z, \infty) \ni y \mapsto \tau(y)\) is right-continuous and non-decreasing, defines a process \(Z \in A(z)\) via [3]. Hence there is a one-to-one correspondence between \(A(z)\) and \(T(z)\), the set of all such families of stopping times. We remark that if a process \(Z \in A(z)\) is bounded from above, say by \(c\), then the process \(Z\) is encoded by the subfamily \((\tau(y))_{z \leq y \leq c}\).

We will frequently use the following change of variable formula.

**Lemma 2.2.** For all measurable functions \(f : [0, T] \to \mathbb{R}_+\) we have

\[
\int_{[0, T]} f(t) dZ_t = \int_{z}^{Z_T} f(\tau(y)) dy.
\]

**Proof.** We first show the result for indicator functions of the form \(f = 1_{[0, t]}\), \(t \in [0, T]\). Note that

\[
\int_{[0, T]} 1_{[0, t]}(s) dZ_s = Z_t - z.
\]
Moreover, since \( \tau(y) \leq t \) if and only if \( Z_t \geq y \),
\[
\int_z^{Z_T} 1_{[0,t]}(\tau(y)) \, dy = \int_z^{Z_t} dy = Z_t - z.
\]
The result follows now by a straightforward monotone class argument. \( \square \)

### 2.2. Optimal primary position via optimal stopping

By Lemma 2.1 any optimal cross hedge is the minimum of a non-decreasing process \( I \) and the weighted primary position \(-hX_t\). In this section we determine the optimal \( X \) for a given process \( I \). Throughout this subsection we fix \( I \in \mathcal{A}(0) \). For any \( X \in \mathcal{A}_0(x_0) \) we define an associated cross hedge \( Y(X)_t = I_t \wedge -hX_t, \ t \in [0,T] \). We will usually omit the dependence on \( X \) and simply write \( Y \) for \( Y(X) \). Notice that \( f(X_t, Y_t) = \tilde{f}(\omega, t, X_t) \), where
\[
\tilde{f}(\omega, t, x) = \begin{cases} (1 - \rho^2)\sigma_1^2 x^2 & \text{if } -hx \leq I_t(\omega), \\ \sigma_2^2 x^2 - 2\rho\sigma_1\sigma_2 I_t(\omega)x + \sigma_2^2 I_t^2(\omega) & \text{else}. \end{cases}
\]
We can formulate the problem of finding an optimal \( X \) for the given process \( I \) as follows:
\[
\inf_{X \in \mathcal{A}_0(x_0)} \mathbb{E} \left[ \int_{[0,T]} K_s dX_s + L \int_{[0,T]} |dI_s \wedge -hX_s| + \int_0^T g'(\tilde{f}(s, X_s)) \tilde{f}_x(s, X_s) \, ds \right]. \tag{4}
\]

The next proposition shows how to derive an optimal primary position path \( X \) from the solutions of a family of stopping problems.

**Proposition 2.3.** Let \( I \in \mathcal{A}(0) \) and fix the cross hedge \( Y_t = I_t \wedge -hX_t \). For all \( x \in [x_0, 0] \) let \( \tau(x) \) be a solution of the stopping problem
\[
\inf_{\tau \in [0,T]} \mathbb{E} \left[ 2Lh1_{\{\tau > \tau'(\cdot\cdot\cdothx)\}} + K_\tau - \int_0^\tau g'(\tilde{f}(s, x)) \tilde{f}_x(s, x) \, ds \right] \tag{5}
\]
such that \( (\tau(x)) \in \mathcal{T}(x_0) \). Then the process \( X \) given by
\[
X_t = \inf\{x \in [x_0, 0] | \tau(x) > t\} \wedge 0
\]
is optimal for \( (4) \).

**Remark 2.4.** The assumption in Proposition 2.3 that \( (\tau(x)) \in \mathcal{T}(x_0) \) is not restrictive. One can show that if there exist stopping times solving the stopping problems (5), then they can be chosen in such a way that they belong to \( \mathcal{T}(x_0) \).

**Remark 2.5.** We give the following interpretation of the stopping problems (5). Instead of finding the optimal entire position path \( X \) in the first place, we may also answer the question of when to buy the infinitesimal unit \( dx \) located at \( x \in [x_0, 0] \) and concatenate the position path afterwards. Determining this optimal point in time \( \tau(x) \) means to find an optimal tradeoff between three terms. Firstly, the term \( K_\tau \) which represents
the marginal costs for buying one unit. Secondly, the integral \( - \int_0^\tau g'(\tilde{f}(s,x))\tilde{f}_x(s,x)ds \) which accounts for risk savings: the sooner the unit located at \( x \) is bought, the smaller the marginal risk it contributes to the aggregate risk, since \( g'(\tilde{f}(s,x))\tilde{f}_x(s,x) \geq 0 \). Finally, the term \( 2Lh1_{\{\tau > \tau^I(-hx)\}} \) which represents the costs incurred in the proxy position: If the unit at \( x \) is cross hedged \( (\tau > \tau^I(-hx)) \) we need to account for the costs of \( 2Lh \); else there are no cross hedging costs.

**Proof.** Let \((\tau^I(y))_{y \geq 0}\) be the family of stopping times associated to \( I \) and let \((\tau(x))_{x \geq x_0}\) be the family encoding a process \( X \in A_0(x_0) \). The change of variable formula of Lemma 2.2 implies that the costs in the primary asset satisfy

\[
\int_{[0,T]} K_s dX_s = \int_{x_0}^0 K_{\tau(x)} dx. \tag{6}
\]

Observe next that \( \max_{s \in [0,T]}(I_s \wedge -hX_s) = \sup\{y \geq 0 | \tau(-\frac{y}{h}) > \tau^I(y)\} \). Hence the execution costs in the secondary asset are given by

\[
L \int_{[0,T]} |dI_s \wedge -hX_s| = 2L \int_{0}^{-x_0/h} 1_{\{\tau(-\frac{y}{h}) > \tau^I(y)\}} dy = 2Lh \int_{x_0}^{0} 1_{\{\tau(x) > \tau^I(-hx)\}} dx.
\]

The risk term satisfies

\[
\int_0^T g(\tilde{f}(s,X_s))ds = -\int_0^T \left( \int_0^{\tau(s)} g'(\tilde{f}(s,x))\tilde{f}_x(s,x)dx - g(\tilde{f}(s,0)) \right) ds \tag{7}
\]

\[
= -\int_0^T \int_{x_0}^{\tau(x)} g'(\tilde{f}(s,x))\tilde{f}_x(s,x)dsdx + \int_0^T g(\tilde{f}(s,0))ds.
\]

The previous calculations show that we can write the sum of time integrals in the expectation of (4) as an integral with respect to the position variable \( x \). The functional in (4) is minimized if and only if the associated stopping times are optimal in (5). \( \square \)

### 2.3. Optimal cross hedges via optimal stopping

The previous subsection describes how to derive optimal primary positions for a given proxy process. In this subsection we consider the opposite problem: for a given primary position process we characterize the optimal cross hedge.

Throughout let \( X \) be a fixed primary asset position process. We can formulate the problem of finding an optimal \( Y \) as follows:

\[
\inf_{I \in A(0)} E \left[ \int_{[0,T]} K_s dX_s + L \int_{[0,T]} |dI_s \wedge -hX_s| + \int_0^T g(\tilde{f}(s,X_s))ds \right] \tag{8}
\]

Problem (8) can again be reduced to a family of stopping problems.
Proposition 2.6. Let $\bar{y} = -hx_0$. For all $y \in [0, \bar{y}]$ let $\tau(y)$ be the solution of the stopping problem

$$
\inf_{\tau \in [0,T]} E \left[ \int_0^T g'(f(X_s, y))[f_y(X_s, y)]^-ds - 2L1_{\{\tau=T\}} \right] \quad (9)
$$

such that $(\tau(y)) \in \mathcal{T}(0)$. Then a cross hedging strategy $Y$ for which the infimum in (8) is attained is given by

$$
Y_t = I_t \wedge -hX_t,
$$

where the process $I$ is the right continuous inverse of $\tau(y)$, i.e. $I_t = \inf\{y \in [0, \bar{y})|\tau(y) > t\}$. 

Proof. Let $(\tau(y))_{y \geq 0} = (\tau^t(y))_{y \geq 0} \in \mathcal{T}(0)$ be the family of stopping times which are optimal in (9). By $(\tau^X(x))_{x \geq x_0}$ we denote the family of stopping times encoding $X$. Then $\tau(y)$ is also optimal in the problem

$$
\inf_{\tau \in [0,T]} E \left[ 1_{\{\tau < \tau^X(-\frac{y}{h})\}} \left( 2L + \int_{\tau}^{\tau^X(-\frac{y}{h})} g'(f(X_s, y))f_y(X_s, y)ds \right) \right] \quad (10)
$$

such that $\tau \in \mathcal{T}(0)$. Indeed, (10) can be rearranged as follows

$$
E \left[ 1_{\{\tau < \tau^X(-\frac{y}{h})\}} \left( 2L + \int_{\tau}^{\tau^X(-\frac{y}{h})} g'(f(X_s, y))f_y(X_s, y)ds \right) \right] = E \left[ 1_{\{\tau < \tau^X(-\frac{y}{h})\}} \left( 2L - \int_{\tau}^{T} g'(f(X_s, y))[f_y(X_s, y)]^-ds \right) \right] = 2L - E \left[ \int_0^T g'(f(X_s, y))[f_y(X_s, y)]^-ds \right] + E \left[ \int_0^T g'(f(X_s, y))[f_y(X_s, y)]^-ds - 2L1_{\{\tau \geq \tau^X(-\frac{y}{h})\}} \right],
$$

where we used the fact that $t \leq \tau^X(-\frac{y}{h})$ is equivalent to $f_y(X_t, y) \leq 0$. Hence, (10) is equivalent to

$$
\inf_{\tau \in [0,T]} E \left[ \int_0^T g'(f(X_s, y))[f_y(X_s, y)]^-ds - 2L1_{\{\tau \geq \tau^X(-\frac{y}{h})\}} \right]. \quad (11)
$$

Since $f_y(X_t, y) \geq 0$ for $t \geq \tau^X(-\frac{y}{h})$, we can restrict ourselves to stopping times taking values in $[0, \tau^X(-\frac{y}{h})) \cup \{T\}$, which implies that $\tau(y)$ is optimal in (11).

Next, define $I_t = \inf\{y \in [0, \bar{y})|\tau(y) > t\}$ and $Y_t = I_t \wedge -hX_t$. Then we have

$$
L \int_{[0,T]} |dY_s| = 2L \int_0^{\bar{y}} 1_{\{\tau(y) < \tau^X(-\frac{y}{h})\}}dy.
$$

11
The risk term of the objective functional can be represented as follows:

\[
\begin{align*}
\int_0^T g(f(X_s, Y_s))ds &= \int_0^T \left( \int_0^{Y_s} g'(f(X_s, y))f_y(X_s, y)dy + g(f(X_s, 0)) \right) ds \\
&= \int_0^T \int_{\tau(y)}^{\tau(y)(-\frac{y}{h})} g'(f(X_s, y))f_y(X_s, y)dy ds 1_{\{\tau(y) < \tau(x)(-\frac{y}{h})\}}dy \\
&\quad + \int_0^T g(f(X_s, 0))ds.
\end{align*}
\]

Hence, the objective functional function is given by

\[
J(0) = E \left[ \int_0^\tau \left( 2L 1_{\{\tau(y) < \tau(x)(-\frac{y}{h})\}} + \int_{\tau(y)}^{\tau(x)(-\frac{y}{h})} g'(f(X_s, y))f_y(X_s, y)dy 1_{\{\tau(y) < \tau(x)(-\frac{y}{h})\}} \right) dy \\
+ \int_0^T g(f(X_s, 0))ds + \int_{[0,T]} K_s ds \right].
\]

Since \(\tau(y)\) is optimal in (10), we obtain that \(I\) is optimal as well.

In the next sections we provide sufficient conditions guaranteeing the optimal cross hedging strategy \(Y\) to be non-increasing after a possible jump at time 0, which means that \(I_t\) is constant. The following example shows that this is not true in general.

**Example 2.1.** Suppose you have a short position of \(x_0 < 0\) in the primary asset and you ask a counterparty to make a sell offer at a price that, in your view, includes no liquidity premium. The counterparty is indecisive about whether to make the offer and asks for some time for consideration. Do you cross hedge until the decision? The following example shows that if you think that the counterparty will accept with a high probability, then you do not cross hedge. It also shows that a cross hedging process \(Y\) is not necessarily non-increasing after time 0.

Consider a non-increasing cost process \(K\) that takes only two values \(K_+\) and \(K_-\), where \(K_+ > K_-\). For simplicity we assume \(K_- = 0\). Suppose that at a deterministic time \(\delta \in (0, T)\) the process jumps from \(K_+\) to the lower level \(K_-\) with probability \(p \in (0, 1)\). With probability \(1 - p\) the process stays constant equal to \(K_+\) and jumps to the lower level only at \(T\). There are only two scenarios for the cost process, and the scenario the process takes is revealed to the agent at \(\delta\).

We suppose that \(\sigma_1 = \sigma_2 = \sigma > 0\), that \(g(x) = \sqrt{x}\) and that

\[
\sigma^2(1 - \rho^2)T + 2L\rho < K_+.
\]

The latter condition implies that it is not optimal to buy a unit of the primary asset before the cost process jumps. The optimal primary asset position is given by

\[
X_t = \begin{cases} 
  x_0 & \text{if } K_t = K_+, \\
  0 & \text{if } K_t = K_-.
\end{cases}
\]
for \( t \in [0,T] \). We suppose that the cross hedging costs \( L \) are low in comparison to the risk entailed by keeping the position open over the whole trading period. More precisely, assume that
\[
2L < \sigma(T - \delta) \quad \text{and} \quad 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{\sigma^2(T - \delta)^2 - 4L^2}} < \rho.
\] (13)

Condition (13) guarantees that
\[
y_2 = \left( \rho - 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{\sigma^2(T - \delta)^2 - 4L^2}} \right) (-x_0)
\] is positive. By Proposition 4.2 it is optimal to cross hedge with a position of \( y_2 \) between \( \delta \) and \( T \) if there has been no jump at \( \delta \). Let \( Y \) be the strategy that is constant equal to \( y_2 \) on \( [0, \delta) \) and constant equal to \( y_2 \) on \( [\delta, T) \) if there has been no jump. Define
\[
A(y) = E \left[ \int_{[0,T]} K_s |dX_s| + \int_{[0,T]} L |dY_s| + \int_0^T \sqrt{f(X_s, y_2)} ds \right]
\]
and observe that
\[
A(y) = 2Lpy + 2L(1-p)y_2 + \delta \sqrt{f(x_0, y)} + (T - \delta)(1-p)\sqrt{f(x_0, y_2)}.
\]

Now suppose that
\[
p > \frac{\delta \sigma}{2L} \in (0, 1).
\] (14)

Then the derivative \( \frac{\partial A}{\partial y} \) is non-positive on \( [0, -\rho x_0] \), which implies that the minimum of \( A(y) \) on \( [0, -\rho x_0] \) is attained at \( y = 0 \). This shows that it is optimal not to build up any cross hedge before \( \delta \).

For the parameters \( K^+ = 1, L = 0.1, T = 1, \sigma = 1, \rho = 0.9 \) and \( p = 0.9 \) the conditions (12), (13) and (14) are satisfied.

2.4. Successive determination of optimal position paths

In many examples the optimal cross hedging process is decreasing after time 0. In such a case one can use an iterative procedure for determining optimal positions. In this subsection we describe this procedure. In the following sections we apply it to specific examples.

Assume that the optimal cross hedge \( Y(X) \) associated to any \( X \in A_0(x_0) \) is non-increasing after 0. Then the value function (2) satisfies
\[
v(x_0) = \inf_{X \in A_0(x_0)} \inf_{y \geq 0} \left[ \int_{[0,T]} K_s dX_s + 2Ly + \int_0^T g(f(X_s, y \wedge hX_s)) ds \right]
\]
\[
= \inf_{y \geq 0} \left( 2Ly + \inf_{X \in A_0(x_0)} E \left[ \int_{[0,T]} K_s dX_s + \int_0^T g(\bar{f}(X_s, y)) ds \right] \right),
\] (15)
with
\[
\bar{f}(x, y) = \begin{cases} 
(1 - \rho^2)\sigma_1^2 x^2 & \text{if } -hx \leq y, \\
\sigma_1^2 x^2 - 2\rho\sigma_1\sigma_2 xy + \sigma_2^2 y^2 & \text{else}.
\end{cases}
\]

Suppose that we can solve, for any \(y \geq 0\), the problem
\[
w(x_0, y) := \inf_{X \in A_0(x_0)} E \left[ \int_{[0,T]} K_s dX_s + \int_0^T g(\bar{f}(X_s, y)) ds \right].
\]
(16)

Moreover assume that there exists a \(y^* \geq 0\) for which the infimum in
\[
v(x_0) = \inf_{y \geq 0} \{2Ly + w(x_0, y)\}
\]
is attained. Then the optimal primary asset position is given by the process \(X^*\) that solves (16) for \(y = y^*\); the optimal cross hedge position is given by \(Y^*_t = y^* \wedge -hX^*_t\) for all \(t \in [0, T]\).

The next proposition shows that the optimal solution of the auxiliary problem (16) can again be characterized by a family of stopping times.

**Proposition 2.7.** For all \(x \in [x_0, 0]\) let \(\tau(x)\) be the solution of the stopping problem
\[
\inf_{\tau \in [0,T]} E \left[ K_\tau - \tau g'(\bar{f}(x, y_0))\bar{f}_x(x, y_0) \right].
\]
Then an optimal primary position \(X\) for (16) is given by
\[
X_t = \inf\{x \in [x_0, 0]|\tau(x) > t\} \wedge 0.
\]

**Proof.** A change of variables as performed in Equations (6) and (7) implies the result. \(\square\)

### 3. Case study: Active trading kicks in at a random time

Trading of forwards usually becomes active as soon as the time to the delivery date falls below a certain time threshold. E.g. a month forward may be actively traded during the 3 months before delivery; but not if the delivery date lies more than 3 months ahead. The trading community usually latently agrees upon a time at which they start trading a forward. Acquiring a forward before the active trading period calls for an additional liquidity premia. Once the trading has become active, the additional premia is no longer asked for. Liquidity in this case does not increase uniformly, but comes suddenly.

The precise time when trading of a particular forward becomes active, however, is often not predictable. Traders can have expectations about when active trading starts, but the precise starting date can be random. In this section we assume that liquidity increases at a random time before maturity, at which active trading kicks in and hence turns the forward market liquid. We have an illiquid trading period before the kick-in date and a liquid one afterwards. For simplicity we assume that the liquidity costs \(K\) are constant before respectively after the kick-in date: \(K\) jumps at a random time \(\tilde{\tau} \in [0, T]\)
from a higher level $K_+ > 0$ to a lower level $K_- \in [0, K_+)$. We model $\hat{\tau}$ as the first jump time of an inhomogeneous Poisson process with non-decreasing jump intensity. More precisely, let $\xi$ be a random variable with standard exponential distribution and $\gamma : \mathbb{R}_+ \to [0, \infty]$ a non-decreasing function with $\gamma \neq 0$. Define $\Gamma(t) = \int_0^t \gamma(s)ds$ and $\hat{\tau} = \Gamma^{-1}(\xi)$. Notice that $\hat{\tau}$ has the same distribution as the first jump time of a Poisson process with jump intensity $\gamma(t)$ at time $t$. We assume that $\Gamma(T) = \infty$; hence, $\hat{\tau} \leq T$ almost surely (in other words, there is a period with active trading).

We start by making two observations:

(i) Every optimal liquidation strategy $(X, Y)$ satisfies $X_t = Y_t = 0$ for all $t \geq \hat{\tau}$.

(ii) Since $X$ is adapted to the filtration generated by $K$, we have

$$X_t = x(t)1_{\{t < \hat{\tau}\}}$$

for some deterministic, non-decreasing function $x : [0, T] \to \mathbb{R}_-$. We will first show that the optimal cross hedge is monotone, i.e. it is optimal to build up an initial cross hedge position and then to reduce it simultaneously with the primary asset (Section 3.1). This result allows us to use the iterative procedure from Section 2.4 for calculating the optimal positions. Proposition 2.7 implies that the optimal primary position is of the form $X_t = x^*1_{\{t < \hat{\tau}\}}$, with $x^* \leq 0$ depending on the initial cross hedge position (Section 3.2). We then obtain the optimal initial cross hedge as the position $y^*$ for which the infimum in (17) is obtained. Since the primary is constant up to the jump time $\hat{\tau}$, this implies that the optimal cross hedge is given by $Y_t = y^*1_{\{t < \hat{\tau}\}}$. The optimal positions, depending on the expected jump time, will be explicitly given in the case where the risk is measured with $g(x) = \lambda \sqrt{x}$; in this case, $\lambda \geq 0$ can be interpreted as a risk aversion parameter (Section 3.3). The proofs of the results of this section are presented in Appendix A.

3.1. Optimal cross hedges are static

In this subsection we show that the stopping times $\tau(y)$ solving the stopping problem (9) are either constant equal to 0 or equal to $T$. In view of Proposition 2.6, this means that the optimal cross hedge for any given primary position process $X \in A_0(x_0)$ is non-increasing after time 0. We can thus use the iterative procedure described in Subsection 2.4 for determining optimal strategies.

**Proposition 3.1.** Let $X \in A_0(x_0)$ be a primary position path. Then for every $y \in [0, -hx_0]$ there exists an optimal stopping time $\tau(y)$ for (9) that takes values only in $\{0, T\}$. In particular, there exists an optimal cross hedging strategy $Y$ of the form $Y_t = y^* \wedge -hX_t$ for some $y^* \in [0, -hx_0]$ attaining the infimum in (8).
3.2. Optimal primary position paths are static

Proposition 3.1 implies that optimal cross hedging strategies are of the form \( Y_t = y \wedge -hX_t \) for some \( y \in [0, -hx_0] \). We define \( Y_0 = y \). We can use Proposition 2.7 to obtain optimal primary position paths from solving the following family of stopping problems

\[
\inf_{\tau \in [0,T]} E [K_\tau - \tau \alpha(x)], \tag{19}
\]

where \( \alpha(x) = g'(\tilde{f}(x,y))\tilde{f}_x(x,y) \). We make the following two observations:

(i) Every optimal stopping time fulfills \( \tau \leq \tilde{\tau} \) almost surely (else the stopping time \( \tau' := \tau \wedge \tilde{\tau} \) performs strictly better since \( \alpha(x) \leq 0 \).

(ii) Let \( \tau \) be a stopping time such that \( \tau \leq \tilde{\tau} \). For measurability reasons there exists a time \( t \in [0, T] \) such that \( \tau = \tau_t := t \wedge \tilde{\tau} \).

The next proposition describes the stopping time solving (19). To this end define

\[
x = \max \left\{ x \leq 0 | \alpha(x) \leq -\frac{E[K_\tau - K_{\tilde{\tau}}]}{E[\tilde{\tau}]} \right\}.
\]

**Proposition 3.2.** Let \( \tau(x) = 0 \) for \( x \in (-\infty, x) \) and \( \tau(x) = \tilde{\tau} \) for \( x \in [x, 0] \). Then \( \tau(x) \) is an optimal stopping time for (19).

3.3. Explicit optimal strategies

In this subsection we derive explicit position paths for a specific choice of the risk function. More precisely, we choose \( g(f(x,y)) = \lambda \sqrt{f(x,y)} \) with \( \lambda \geq 0 \). This corresponds to a first order approximation of the position’s value at risk. From the preceding sections we know that optimal strategies are static, i.e. \( X^*_t = x^*_1 \{ t < \tilde{\tau} \} \) and \( Y^*_t = y^*_1 \{ t < \tilde{\tau} \} \). We will derive explicit formulas for \( x^* \) and \( y^* \) in terms of the model parameters. To this end we distinguish several cases.

**Proposition 3.3.** Let \( \Delta K = K_0 - E[K_{\tilde{\tau}}] \) and

\[
M = \frac{\sigma_0}{\sigma_1} \left( \Delta K \rho - \sqrt{(1 - \rho^2)(\lambda^2 \sigma^2 E[\tilde{\tau}]^2 - \Delta K^2)}^+ \right).
\]

1. If \( \Delta K \leq \lambda \sigma_1 \sqrt{1 - \rho^2} E[\tilde{\tau}] \), then it is optimal to close the primary position immediately and not to cross hedge, i.e. \( x^* = y^* = 0 \).

2. If \( \Delta K \geq \lambda \sigma_1 E[\tilde{\tau}] \), then it is optimal to keep the primary position open and to hedge with \( y^* = -\frac{\sigma_1}{\sigma_2} \max \left( 0, \rho - 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{\lambda^2 \sigma^2 E[\tilde{\tau}]^2 - 4L^2}} \right) x_0 \) units of the proxy.

3. Suppose that \( \lambda \sigma_1 \sqrt{1 - \rho^2} E[\tilde{\tau}] \leq \Delta K < \lambda \sigma_1 E[\tilde{\tau}] \). If \( M \leq 2L \), then it is optimal to close the primary position immediately and not to cross hedge. If \( M \geq 2L \), then it is optimal to keep the primary position open and to hedge with \( y^* = -\frac{\sigma_1}{\sigma_2} \left( \rho - 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{\lambda^2 \sigma^2 E[\tilde{\tau}]^2 - 4L^2}} \right) x_0 \) units of the proxy.
4. Case Study: Deterministic liquidity costs

In this section we turn to deterministic continuous liquidity processes $K$ and derive explicit optimal strategies. The optimal position paths are essentially determined by the time decay of liquidity costs.

If the position of the primary and the proxy is constant equal to $(x, y)$ on $[t, T)$, then the associated risk decays linearly in $t$ at a rate of $g(f(x, y))$. The liquidity costs implied by buying one unit of the primary asset decrease at rate $\dot{K}_t$. Suppose now that the initial liquidity costs are high compared to the risk such that it is not optimal to buy a unit of the primary asset at $t = 0$. If $\dot{K}_t$ is non-increasing, then the costs do not decrease faster than linearly. Hence costs remain high relative to the risk and it is optimal not to buy before $T$. In Section 4.1 we confirm this intuition by showing that it is optimal to close the whole position either immediately or at $T$ if $\dot{K}_t$ is non-increasing (i.e. if $K$ is concave). In Section 4.2 we treat the case where $K$ is convex. The proofs of the results of this section are presented in Appendix B.

4.1. Concave decay of liquidity costs

We assume that $K$ is concave and deterministic. Since no randomness is involved in the model set-up, we can restrict ourselves to deterministic execution strategies. First we consider the problem of finding optimal cross hedging strategies for a fixed primary position path $X$. Note that for every $y \leq 0$ the mapping

$$t \mapsto \int_0^t g'(f(X_s, y))[f_y(X_s, y)]^{-1} ds - 2L 1_{\{t=T\}}$$

is non-decreasing on $[0, T)$ with a possible downward jump at time $T$. Hence, it attains its minimum at 0 or $T$. Proposition 2.6 implies that optimal cross hedges are of the form $Y_t = y \wedge -hX_t$ for some $y \geq 0$. Next, we use Proposition 2.7 and consider the stopping problem

$$\inf_{\tau \in [0,T]} K_\tau - t g'(\bar{f}(x, y)) \bar{f}_x(x, y).$$

Concavity of $K$ implies that 0 or $T$ are optimal. Hence, the static path $X_t = 1_{\{t<T\}}x(y)$ with

$$x(y) = \max\{x \leq 0|K_0 \leq K_T - g'(\bar{f}(x, y))\bar{f}_x(x, y)T\}$$

is optimal.

For the specific choice $g(x) = \lambda \sqrt{x}$ we can perform similar calculations as in Section 3.3. The next proposition provides explicit optimal strategies. To simplify notation we define $\Delta K = K_0 - K_T$ and the non-negative number

$$A = -\frac{\sigma_1}{\sigma_2} \max\left(0, \rho - 2L \frac{\sqrt{1-\rho^2}}{\sqrt{\lambda^2 \sigma_2^2 T^2 - 4L^2}}\right) x_0.$$ (20)
Proposition 4.1 (Concave case). Suppose that \( g(x) = \lambda \sqrt{x} \) (\( \lambda \geq 0 \)) and that \( K \) is decreasing and concave on \([0,T]\). Then there exists an optimal strategy that is static, i.e. of the form

\[
X^*_t = x^*1_{[0,T]}(t), \quad Y^*_t = y^*1_{[0,T]}(t),
\]

with \( x^* \leq 0 \) and \( y^* \geq 0 \). The optimal positions are as follows:

(C1) If \( \Delta K \leq \lambda \sigma_1 \sqrt{1 - \rho^2} T \), then it is optimal to close the primary position immediately and not to cross hedge, i.e. \( x^* = y^* = 0 \).

(C2) If \( \Delta K \geq \lambda \sigma_1 T \), then it is optimal to keep the primary position open and to hedge with \( y^* = A \) proxy contracts.

(C3) If \( \lambda \sigma_1 \sqrt{1 - \rho^2} T \leq \Delta K \leq \lambda \sigma_1 T \) and \( \frac{\sigma_2}{\sigma_1} \left( \Delta K \rho - \sqrt{(1 - \rho^2)(\lambda^2 \sigma_1^2 T^2 - \Delta K^2)} \right) \leq 2L \), then it is optimal to close the primary position immediately and not to cross hedge.

(C4) If \( \lambda \sigma_1 \sqrt{1 - \rho^2} T \leq \Delta K \leq \lambda \sigma_1 T \) and \( \frac{\sigma_2}{\sigma_1} \left( \Delta K \rho - \sqrt{(1 - \rho^2)(\lambda^2 \sigma_1^2 T^2 - \Delta K^2)} \right) \geq 2L \), then it is optimal to keep the primary position open and to hedge with \( y^* = A \) proxy contracts.

In the following, we give a brief economic interpretation. In case (C1) the additional liquidity costs from an early closure are smaller than the risk entailed by keeping the primary position open until \( T \). Since the speed of the cost decay is non-decreasing, it can not be optimal to close the position at an intermediate point between 0 and \( T \). It is, therefore, optimal to close the primary position immediately. Consequently, there is no need for a cross hedge.

In case (C2) the additional liquidity costs exceed the risk, even if no hedge is performed. Again it is not optimal to close the position at an intermediate point between 0 and \( T \); hence it is optimal to keep the primary position open until \( T \). The optimal cross hedge position is static, too and can be derived by a straightforward calculation. Notice that if the costs \( L \) for trading the proxy are high, then no cross hedge position is taken.

If the initial liquidity costs lie between \( \rho \sigma_2^2 T \) and \( \sigma_2^2 T \), then the costs for a cross hedge determine whether it is optimal to close the primary position immediately or not. In case (C3) the costs are too high; hence the position is closed at \( t = 0 \), and no cross hedge is performed. In case (C4) the liquidity costs are low; it is optimal to keep the primary position open and to cross hedge.

From Proposition 4.1 we derive a simple decision rule whether to cross hedge or not. It is described in Corollary 4.2 and illustrated in the decision tree in Figure 1.

Corollary 4.2. Let the assumptions of Proposition 4.1 hold true and

\[
\bar{L} = \frac{\sigma_2}{2 \sigma_1} \left( \Delta K \rho - \sqrt{(1 - \rho^2)(\lambda^2 \sigma_1^2 T^2 - \Delta K^2)} \right). \tag{22}
\]

The optimal positions in (21) are as follows:
If $L < \bar{L}$, then it is optimal to keep the primary position open and to cross hedge with $A$ units of the proxy; i.e. $x^* = x_0$ and $y^* = A$.

2. If $L \geq \bar{L}$, then it is optimal not to cross hedge (i.e. $y^* = 0$). Whether it is optimal to immediately close the primary position depends on the size of the cost increment. If $\Delta K \geq \lambda \sigma_1 T$, then $x^* = x_0$. If $\Delta K < \lambda \sigma_1 T$, then $x^* = 0$.

4.2. Convex decay of liquidity costs

If the speed of the cost decay decreases, then it can be reasonable for the agent to postpone the liquidation of the primary position, even if immediate liquidity costs are smaller than the additional risk that this unit entails. Essentially it is optimal to buy as soon as the cost saving rate does not exceed the rate of the additional risk.

In the following we suppose that the liquidity cost process $K$ is strictly convex and $K \in C^1$, which means that the cost saving rate is decreasing in time. Note that this implies that $\dot{K}$ is strictly increasing and hence invertible. Moreover we assume that there are no cross hedging costs: $L = 0$. Given a primary position $X_t$ this implies that it is optimal to perform a minimum variance hedge in the proxy position: $Y_t = -hX_t$ for all $t \in [0,T]$. The optimal stopping problem of Proposition 2.7 then simplifies to

$$\inf_{\tau \in [0,T]} E[K_\tau - \tau G'(x)],$$

(23)

where $G(x) = g((1 - \rho^2)\sigma_1^2 x^2)$ for $x \in [x_0, 0]$. Note that $G$ is convex and continuously differentiable, and $G'(x) \leq 0$ for $x \leq 0$. 

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As no randomness is involved the optimal time solving \(23\) is deterministic and is given by

\[
t(x) = \begin{cases} 
T & \text{if } G'(x) \geq \\ 
(\dot{K})^{-1}(G'(x)) & \text{if } \dot{K}_0 \leq G'(x) < \dot{K}_T, \\
0 & \text{if } G'(x) < \dot{K}_0.
\end{cases}
\]

It is straightforward to show that \(x \mapsto t(x)\) is continuous and non-decreasing on \((-\infty, 0]\).

We set

\[
b = \max\{x \leq 0|G'(x) \leq \dot{K}_0\}, \\
a = \min\{x \leq 0|G'(x) \geq \dot{K}_T\},
\]

where we use the convention \(\min\emptyset = 0\) and \(\max\emptyset = -\infty\). Note that \(0 \geq a \geq b \geq -\infty\). Proposition \(2.3\) implies that the optimal primary asset position trajectory can be recovered as the inverse of the mapping \(t(x)\). The next proposition describes the optimal strategy precisely.

**Proposition 4.3.** Suppose that \(L = 0\) and that \(K\) is decreasing, continuously differentiable and strictly convex on \([0, T]\). The optimal primary position strategy \((X_t)_{t \in [0, T]}\) of closing \(x_0 < 0\) is given as follows:

1. At time \(t = 0\) it is optimal to buy the amount of \((b - x_0)_+\), i.e. \(X_0 = \max\{x_0, b\}\).
2. The position is continuously increased between \(t(X_0)\) and \(t(a)\). More precisely, the optimal position at \(t \in [t(X_0), t(a)]\) is given by \(X_t = (G')^{-1}(\dot{K}_t)\).
3. The remaining open position that has to be closed at time \(T\) is given by \(a \lor x_0\) (note that \(a\) may be zero).

The optimal cross hedge position is \(Y_t = -hX_t\).

If the risk is measured with \(g(x) = \lambda \sqrt{x}\), then it is optimal to close the primary position in one go. Notice that in this case the risk is linear in the position size. Also the liquidity costs are proportional to the position size. Therefore, as soon as it is optimal to buy one unit of the primary asset, it is optimal to close the whole primary position immediately.

**Corollary 4.4** (Strict convex case). Suppose that the assumptions of Proposition 4.3 hold true and that \(g(x) = \lambda \sqrt{x}\).

1. If \(\dot{K}(T) < -\lambda \sigma_1 \sqrt{1 - \rho^2}\), then the optimal position processes are given by \(X^* = x_0 1_{[0, T]}\) and \(Y^* = -hx_0 1_{[0, T]}\).
2. If \(\dot{K}(0) > -\lambda \sigma_1 \sqrt{1 - \rho^2}\), then the optimal position processes are given by \(X^* = Y^* = 0\).
3. If $\lambda \sigma \sqrt{1 - \rho^2} \in [-K(T), -K(0)]$, then the optimal buying time is given by

$$t^* = (\dot{K})^{-1}(-\lambda \sigma \sqrt{1 - \rho^2}),$$

and $X^* = x_0 1_{[0, t^*)}$ and $Y^* = -hx_0 1_{[0, t^*)}$ are the optimal position processes.

**Proof.** Notice that in this case the derivative $G'$ is constant equal to $\lambda \sigma \sqrt{1 - \rho^2}$. In particular, the points $a$ and $b$ are either equal to 0 or to $-\infty$. The result now follows directly from Proposition 4.3. □

**Appendix**

**A. Proofs of the results of Section 3**

For the proof of Proposition 3.1, we first require some preliminary considerations. We fix a primary position process $X_t = x(t) 1_{\{t < \tilde{\tau}\}}$ as in (18). For ease of notation we introduce the non-negative, non-increasing function

$$\phi(t) = \left[f_y(x(t), y)\right] - g'(f(x(t), y))$$

and the process

$$A_t = \int_0^t \phi(s) 1_{\{s < \tilde{\tau}\}} ds - 2L 1_{\{t = T\}},$$

where we suppress the dependence on $X$ and $y$. Then the stopping problem (9) can be rewritten as

$$\inf_{\tau \in [0, T]} E[A_\tau]. \tag{24}$$

Furthermore we introduce the mapping

$$\beta(t) = \int_t^T \phi(s) P(s < \tilde{\tau} | t < \tilde{\tau}) ds.$$  

Notice that $\beta$ is non-increasing. Indeed, we have $\beta(t) = e^{\Gamma(t)} \int_t^T \phi(s) e^{-\Gamma(s)} ds$, and hence the monotonicity of $\gamma$ and $\phi$ imply

$$\beta'(t) = \gamma(t) e^{\Gamma(t)} \int_t^T \phi(s) e^{-\Gamma(s)} ds - \phi(t) \leq \phi(t) \left(e^{\Gamma(t)} \int_t^T \gamma(s) e^{-\Gamma(s)} ds - 1\right) = -\phi(t) e^{-(\Gamma(T) - \Gamma(t))} \leq 0.$$  

We proceed with two auxiliary lemmas.
Lemma A.1. Assume that $\beta(t) \leq 2L$ for all $t \in [0,T]$. Then $\tau = T$ is an optimal stopping time of $(24)$.

Proof. For $s \geq t$ we have

$$E[1_{\{s<\tilde{\tau}\}}|\mathcal{F}_t] = 1_{\{t<\tilde{\tau}\}}P(s < \tilde{\tau}|t < \tilde{\tau}).$$

Thus we obtain for all $t \in [0,T]$

$$E[A_T - A_t|\mathcal{F}_t] = \int_t^T \phi(s)E[1_{\{s<\tilde{\tau}\}}|\mathcal{F}_t]ds - 2L = 1_{\{t<\tilde{\tau}\}}\beta(t) - 2L \leq 0.$$ 

This implies that the Snell envelope $U$ of $A$ is given by $U_t = E[A_T|\mathcal{F}_t]$ and that $\tau = T$ is optimal.

Lemma A.2. Let $a = \inf\{t \geq 0|\beta(t) \leq 2L\} \leq T$. Then we have for all $t \leq r \leq a$

$$\int_t^r \phi(s)P(s < \tilde{\tau}|t < \tilde{\tau})ds \geq 2LP(\tilde{\tau} \leq r|\tilde{\tau} > t). \tag{25}$$

Proof. Note first that (25) is equivalent to

$$\int_t^r \phi(s)P(s < \tilde{\tau})ds \geq 2LP(t < \tilde{\tau} \leq r).$$

Since $\beta$ is non-increasing we have

$$\int_t^r \phi(s)P(s < \tilde{\tau})ds = P(t < \tilde{\tau})\beta(t) - P(r < \tilde{\tau})\beta(r) \geq P(t < \tilde{\tau} \leq r)\beta(r) \geq 2LP(t < \tilde{\tau} \leq r),$$

which completes the proof.

Proof of Proposition 3.1. If $\beta(0) < 2L$, then Lemma A.1 implies that $\tau^* = T$ is an optimal stopping time.

For the rest of the proof assume that $\beta(0) \geq 2L$. Let $N \in \mathbb{N}$ and

$$\Delta = \{0 = t_0 < t_1 < \cdots < t_N = a\}$$

be a finite partition of $[0,a]$, where $a = \inf\{t \geq 0|\beta(t) \leq 2L\} \leq T$. We denote by $(U^\Delta_{t_i})_{0 \leq t_i \leq T}$ the Snell envelope of the stopping problem

$$\inf_{\tau \in \Delta \cup [a,T]} E[A_\tau].$$

We write $U^\Delta_t = U^\Delta_{t_i}$. Then by Lemma A.1

$$U^\Delta_N = U^\Delta_a = E[A_T|\mathcal{F}_a],$$

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and by definition for $0 \leq i \leq N - 1$

$$U_i^\Delta = E[U_{i+1}^\Delta | \mathcal{F}_t] \land A_{t_i}.$$  

We next show that

$$U_i^\Delta = A_{t_i} \mathbf{1}_{\{\tilde{\tau} > t_i\}} + E[A_T | \mathcal{F}_t] \mathbf{1}_{\{\tilde{\tau} \leq t_i\}}$$  \hspace{1cm} (26)

for all $0 \leq i \leq N$. In particular, this implies $U_0^\Delta = A_0$. Hence, $\tau^* = 0$ is optimal among all stopping times taking values in $\Delta \cup [a, T]$. We prove (26) by backwards induction.

For $i = N$ we have

$$E[A_T - A_a | \mathcal{F}_a] = -2L1_{\{\tilde{\tau} \leq a\}}.$$  

Hence

$$E[A_T | \mathcal{F}_a] = A_a$$  \hspace{1cm} on $\{\tilde{\tau} > a\}$, which implies (26). Let now $0 \leq i \leq N - 1$. On $\{\tilde{\tau} \leq t_i\}$ we have $\{\tilde{\tau} \leq t_{i+1}\}$, which implies

$$E[U_{i+1}^\Delta - A_{t_i} | \mathcal{F}_t] = E[A_T - A_{t_i} | \mathcal{F}_t] = -L < 0.$$  

Hence, $U_i^\Delta = E[A_T | \mathcal{F}_t]$ on $\{\tilde{\tau} \leq t_i\}$. On $\{\tilde{\tau} > t_i\}$ we have

$$U_{i+1}^\Delta = A_{t_i} - 2L1_{\{\tilde{\tau} \leq t_{i+1}\}} = A_{t_i} + \int_{t_i}^{t_{i+1}} \phi(s) \mathbf{1}_{\{s < \tilde{\tau}\}} ds - 2L1_{\{\tilde{\tau} \leq t_{i+1}\}}.$$  

This implies on $\{\tilde{\tau} > t_i\}$

$$E[U_{i+1}^\Delta | \mathcal{F}_{t_{i+1}}] = A_{t_i} + \int_{t_i}^{t_{i+1}} \phi(s) P(s < \tilde{\tau} | t_i < \tilde{\tau}) ds - 2LP(\tilde{\tau} \leq t_{i+1} | \tilde{\tau} > t_i).$$

Equation (26) now follows from Lemma A.2.

It remains to show that $\tau^* = 0$ is also optimal among all stopping times taking values in $[0, T]$. Let $\tau$ be such a stopping and define

$$\tau^N = \begin{cases} \frac{k}{N}a & \text{if } \tau \in (\frac{k}{N-1}a, \frac{k}{N}a) \text{ for a } 1 \leq k \leq N, \\ \tau & \text{else.} \end{cases}$$

Notice that $\lim_{N \to \infty} E[A_{\tau^N} | \mathcal{F}_t] = E[A_T | \mathcal{F}_t]$. The fact that $\tau \neq 0$ a.s. and $E[A_{\tau}] < E[A_0]$ yield a contradiction to the optimality of $\tau^* = 0$ among the stopping times taking only finitely many values on $[0, a]$.

Next we provide the proof of Proposition 3.2.

**Proof of Proposition 3.2** Fix $x \leq 0$ and consider

$$\phi(t) := E[K_{\tau_t} - \alpha \tau_t] = K_+ P(\tilde{\tau} > t) + K_- P(\tilde{\tau} \leq t) - \alpha \left( E[\tilde{\tau} 1_{\{\tilde{\tau} \leq t\}}] + tP(\tilde{\tau} > t) \right)$$

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on $[0,T]$. We readily compute
\[ P[\tilde{\tau} \leq t] = P[\Gamma^{-1}(\xi) \in [0, t]] = 1 - \exp(-\Gamma(t_0)) \]
and
\[ P[\tilde{\tau} > t] = \exp(-\Gamma(t)) \]
Finally,
\[ E[\tilde{\tau} \mathbf{1}_{\{\tilde{\tau} \leq t\}}] = \int \Gamma^{-1}(\xi) \mathbf{1}_{[0,t]}(\Gamma^{-1}(\xi))dP = \int_0^{\Gamma(t)} \Gamma^{-1}(s) \exp(-s)ds. \]
Hence we have
\[ \phi'(t) = \exp(-\Gamma(t))(-\alpha - \gamma(t)(K_+ - K_-)) \]
and \( \phi'(t) = 0 \) if and only if \( -\alpha = \gamma(t)(K_+ - K_-) \).
Furthermore,
\[ \phi''(t) = -\gamma(t)\phi'(t) + \exp(-\Gamma(t))(-\gamma'(t)(K_+ - K_-)). \]
Therefore, if \( \gamma \) is strictly increasing on \([0,T]\), the unique local extremum of \( \phi \) is a maximum. Hence, \( \phi(t) \) attains its minimum at 0 or \( T \).
Notice that \( \alpha \) is non-decreasing on \( \mathbb{R}_- \). Therefore, for \( x \leq \bar{x} \) the minimum is attained at \( t = 0 \), and for \( x > \bar{x} \) at \( t = T \).

Let us now turn to the proof of Proposition 3.3. We introduce the function \( H(x, y) \):
\[ H(x, y) = \lambda \sqrt{f(x, y)} = \begin{cases} -\lambda \sigma_1 x \sqrt{1 - \rho^2} & \text{if } y \geq -hx, \\ \lambda \sqrt{\sigma_1^2 x^2 + 2\rho \sigma_1 \sigma_2 x y + \sigma_2^2 y^2} & \text{if } y < -hx. \end{cases} \]
By Proposition 3.1, cross hedging strategies are non-increasing after a possible jump at time 0. Hence, Equation (15) yields that the value function is given by
\[ v(x_0) = \inf_{y \geq 0} w(x_0, y) + 2Ly \]
with \( w(x_0, y) \) defined as in (16). Proposition 3.2 implies that \( w \) is given by
\[ w(x_0, y) = K_0(x(y) - x_0)_+ - E[K_{\tau^T}] \max(x_0, x(y)) + E[\tilde{\tau}]H(\max(x_0, x(y)), y) \]
with
\[ x(y) = \max\{x \leq 0 | K_0 \leq E[K_{\tau^T}] - H_x(x, y)E[\tilde{\tau} \wedge T]\}, \tag{27} \]
and that the infimum of \( w \) is attained for the position process \( X_t = (x(y) \vee x_0)1_{t<\tilde{\tau}} \). The proof of Proposition 3.3 consists of computing \( x(y) \) and \( w(x_0, y) \) explicitly for the particular cases and determining \( y^* \geq 0 \) satisfying \( v(x_0) = w(x_0, y^*) + 2Ly^* \) afterwards.
Proof of Proposition 3.3. Before proving the statements notice that
\[ H_x(x, y) = \begin{cases} -\lambda \sigma_1 \sqrt{1 - \rho^2} & \text{if } y \geq -hx, \\ \frac{\lambda}{\sqrt{\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 y + \sigma_2^2 y^2}} & \text{if } y < -hx. \end{cases} \]

We now show the three statements separately.

1. Since \( H_x \) is bounded from above by \( -\lambda \sigma_1 \sqrt{1 - \rho^2} \), we have \( x(y) = 0 \). Hence, \( x^* = 0 \). This implies \( w(x_0, y) = -K_0 x_0 \) and \( y^* = 0 \).

2. Since \( H_x \) is bounded from below by \( -\lambda \sigma_1 \), we have \( x(y) = -\infty \), which implies \( x^* = x_0 \). Hence, we have \( w(x_0, y) = -E[K_{\tilde{x}}] x_0 + E[\tilde{x}] H(x_0, y) \). Note that

\[ H_y(x_0, y) = \frac{\lambda^2 \sigma_2^2}{2 \sigma_2^2 x_0^2 + 2 \rho \sigma_1 \sigma_2 x_0 y + \sigma_2^2 y^2} x_0 + y \]

for \( y < -hx_0 \), which is increasing in \( y \) with \( H_y(x_0, -hx_0) = 0 \) and \( H_y(x_0, 0) = -\lambda \rho \sigma_2 \). So, if \( 2L \geq \lambda \sigma_2 \rho E[\tilde{x}] \), then \( w(x_0, y) + 2Ly \) attains its minimum at \( y^* = 0 \). Else, \( y^* \) is the solution of \( E[\tilde{x}] H_y(x_0, y^*) = -2L \) on \([0, -hx_0]\), i.e.

\[ y^* = -\frac{\sigma_1}{\sigma_2} \left( \rho - 2L \sqrt{\frac{1 - \rho^2}{\lambda^2 \sigma_2^4 E[\tilde{x}]^2 - 4L^2}} \right) x_0. \]

3. Note that by Equation (27), \( x(y) \) is given implicitly by the solution of \( \Delta K = -H_x(x(y), y) E[\tilde{x}] \) on \((-\infty, -y/h]\); hence, \( x(y) = -\alpha y \) with

\[ \alpha = \frac{\sigma_2}{\sigma_1} \left( \rho + \frac{\sqrt{1 - \rho^2}}{\sqrt{\lambda^2 \sigma_2^4 E[\tilde{x}]^2 - \Delta K^2}} \right) \in [1/h, \infty). \]

For \( y \in [0, -x_0/\alpha] \) (which is equivalent to \( x_0 \leq x(y) \)) we have

\[ w(x_0, y) + 2Ly = K_0(-x_0 - \alpha y) + \alpha E[K_{\tilde{x}}] y + E[\tilde{x}] H(-\alpha y, y) + 2Ly \]

\[ = -K_0 x_0 + (-\alpha \Delta K + \lambda E[\tilde{x}] \sqrt{\sigma_1^2 \alpha^2 - 2 \rho \alpha \sigma_1 \sigma_2 + \sigma_2^2 + 2L}) y \]

\[ = K_0 x_0 + my, \]

with

\[ m = \frac{\sigma_2}{\sigma_1} \left( -\Delta K \rho + \sqrt{1 - \rho^2} \sqrt{\lambda^2 E[\tilde{x}]^2 \sigma_1^2 - \Delta K^2} \right) + 2L. \]

For \( y \in [-x_0/\alpha, -hx_0] \) (or equivalently \( x_0 \geq x(y) \)) we have

\[ w(x_0, y) + 2Ly = -E[K_{\tilde{x}}] x_0 + E[\tilde{x}] H(x_0, y) + 2Ly \]

\[ = -E[K_{\tilde{x}}] x_0 + \lambda E[\tilde{x}] \sqrt{\sigma_1^2 x_0^2 + 2 \rho \sigma_1 \sigma_2 x_0 y + \sigma_2^2 y^2 + 2Ly}. \]
We verify the following claim at the end of the proof.

**Claim:** The mapping \( y \mapsto w_y(x_0, y) + 2L \) is continuous and non-decreasing on \([0, -hx_0]\). Moreover it is constant equal to \( m \) on \([0, -x_0/\alpha]\) and satisfies

\[
w_y(x_0, -hx_0) + 2L = 2L > 0.
\]

If \( m \geq 0 \), then \( y \mapsto w(x_0, y) + 2Ly \) is non-decreasing on \([0, -hx_0]\). This implies \( y^* = 0 \) as well as \( x^* = x(y^*) = 0 \). Else \( y^* \) is the solution of

\[
E[\tilde{\tau}]H_y(x_0, y^*) = -2L
\]
on \([{-x_0/\alpha, -hx_0}]\). A straightforward calculation yields

\[
y^* = -\frac{\sigma_1}{\sigma_2} \left( \rho - 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{(\lambda^2\sigma_2^2 E[\tilde{\tau}]^2 - 4L^2)^+}} \right) x_0.
\]

Note that \( y^* \geq -x_0/\alpha \) implies \( x(y^*) = -\alpha y^* \leq x_0 \). Hence, we have \( x^* = x_0 \).

It remains to prove the claim. Since \( y \mapsto H(x_0, y) \) is convex, monotonicity follows immediately. It is hence sufficient to show continuity of \( y \mapsto w_y(x_0, y) + 2L \); more precisely,

\[
\lim_{y \searrow -x_0/\alpha} w_y(x_0, y) + 2L = m.
\] (28)

We have

\[
w_y(x_0, y) = \lambda E[\tilde{\tau}] \frac{\rho \sigma_1 \sigma_2 x_0 + \sigma_2^2 y}{\sqrt{\sigma_1^2 x_0^2 - 2\rho \sigma_1 \sigma_2 x_0 y + \sigma_2^2 y^2}} \rightarrow -\lambda E[\tilde{\tau}] \frac{\rho \sigma_1 \sigma_2 \alpha - \sigma_2^2}{\sqrt{\sigma_1^2 \alpha^2 - 2\rho \sigma_1 \sigma_2 \alpha + \sigma_2^2}}
\]
as \( y \searrow -x_0/\alpha \). By its defining property, \( \alpha \) satisfies

\[
\Delta K = -H_x(-\alpha y, y) E[\tilde{\tau}]
\]

\[
= -\lambda E[\tilde{\tau}] \frac{\rho \sigma_1 \sigma_2 - \sigma_2^2 \alpha}{\sqrt{\sigma_1^2 \alpha^2 - 2\rho \sigma_1 \sigma_2 \alpha + \sigma_2^2}}
\]

\[
= \frac{\lambda E[\tilde{\tau}]}{\alpha} \left( \sqrt{\sigma_1^2 \alpha^2 - 2\rho \sigma_1 \sigma_2 \alpha + \sigma_2^2} + \frac{\rho \sigma_1 \sigma_2 \alpha - \sigma_2^2}{\sqrt{\sigma_1^2 \alpha^2 - 2\rho \sigma_1 \sigma_2 \alpha + \sigma_2^2}} \right).
\]

But this is equivalent to

\[
\lambda E[\tilde{\tau}] \frac{\rho \sigma_1 \sigma_2 \alpha - \sigma_2^2}{\sqrt{\sigma_1^2 \alpha^2 - 2\rho \sigma_1 \sigma_2 \alpha + \sigma_2^2}} = \alpha \Delta K - \lambda E[\tilde{\tau}] \sqrt{\sigma_1^2 \alpha^2 - 2\rho \sigma_1 \sigma_2 \alpha + \sigma_2^2} = -m + 2L,
\]

which implies (28).
B. Proofs of the results of Section 4

Proposition 4.1 is obtained by performing similar calculations as in the proof of Proposition 3.3.

Proof of Corollary 4.2. Observe that if $\Delta K \leq \sqrt{1 - \rho^2} \lambda \sigma_1 T$, then $\bar{L} \leq 0$. Since $L \geq 0$, we also have $\bar{L} \leq L$. The following implication, therefore, holds true:

$$L < \bar{L} \implies \Delta K > \sqrt{1 - \rho^2} \lambda \sigma_1 T.$$  (29)

Next we show

$$L \geq \bar{L} \text{ and } \Delta K \geq \lambda \sigma_1 T \implies A = 0.$$  (30)

Assume that $L \geq \bar{L}$ and $\Delta K \geq \lambda \sigma_1 T$. In order to prove (30) it suffices to show that

$$L \geq \frac{\rho}{2} \frac{\sqrt{(\lambda^2 \sigma_2^2 T^2 - 4 L^2)^+}}{\sqrt{1 - \rho^2}}.$$  (31)

Notice that $L \geq \bar{L} \geq \frac{\rho}{2} \lambda \sigma_2 T$. Moreover,

$$\frac{\rho}{2} \frac{\sqrt{(\lambda^2 \sigma_2^2 T^2 - 4 L^2)^+}}{\sqrt{1 - \rho^2}} \leq \frac{\rho}{2} \frac{\sqrt{(\lambda^2 \sigma_2^2 T^2 - \rho^2 \lambda^2 \sigma_2^2 T^2)^+}}{\sqrt{1 - \rho^2}} = \frac{\rho}{2} \lambda \sigma_2 T,$$

which yields Inequality (31).

We now prove the statements of Corollary 4.2. Assume first that $L \geq \bar{L}$. Implication (30) shows that in case (C2) of Proposition 4.1 we have $A = 0$ and hence that $y^* = 0$. It is therefore never optimal to cross hedge in this case. The primary position is kept open if $\Delta K \geq \lambda \sigma_1 T$ (case (C2)). If $\Delta K < \lambda \sigma_1 T$, then it is optimal to close the primary position immediately (cases (C1) and (C3)).

Next assume that $L < \bar{L}$. From Implication (29) we know that in this case $\Delta K > \sqrt{1 - \rho^2} \lambda \sigma_1 T$. Cases (C2) and (C4) further imply that $x^* = x_0$ and $y^* = A$.

References


